

Errors, Robustness, and The Fourth Quadrant

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Abstract: The paper presents evidence that econometric techniques based on variance- L^2 norm are flawed –and do not replicate. The result is un-computability of role of tail events. The paper proposes a methodology to calibrate decisions to the degree (and computability) of forecast error. It classifies decision payoffs in two types: simple payoffs (true/false or binary) and complex (higher moments); and randomness into type-1 (thin tails) and type-2 (true fat tails) and shows the errors for the estimation of small probability payoffs for type 2 randomness. The Fourth Quadrant is where payoffs are complex with type-2 randomness. We propose solutions to mitigate the effect of the Fourth Quadrant based on the nature of complex systems.

Keywords: complexity, decision theory, fat tails, risk management

Second Draft

I- BACKGROUND AND PURPOSE¹

It appears scandalous that, of the hundreds of thousands of professionals involved, including prime public institutions such as the World Bank, the International Monetary Fund, different governmental agencies and central banks, private institutions such as banks, insurance companies, and large corporations, and, finally, academic departments, only a few individuals considered the possibility of the total collapse of the banking system that started in 2007 (and is still worsening at the time of writing), as well as the economic consequences of such breakdown. Not a single official forecast turned out to be close to the outcome experienced –even those issuing “warnings” did not come close to the true gravity of the situation. A few warnings about the risks such as Taleb (2007) or the works of the economist Nouriel Roubini² went

unheeded, often ridiculed³. Where did such sophistication go? In the face of such proportion of miscalculation, it would seem fitting to start an examination of the conventional forecasting methods for risky outcomes and assess their fragility –indeed the size of the damage comes from confidence in forecasting and the mis-estimation of potential forecast errors for a certain classes of variables and a certain type of exposures. But this was not the first time such events happened –nor was it a “Black Swan” (when capitalized, an unpredictable outcome of high impact) to the observer who took a close view at the robustness and empirical validity of the methods used in economic forecasting and risk measurement.

This examination, while grounded in economic data, generalizes to all decision-making under uncertainty in which there is a potential miscalculation of the risk of a consequential rare event. The problem of concern is the rare event, and the exposure to it, of the kind that can fool a decision maker into taking a certain course of action based on the misunderstanding of the risks involved.

¹ A longer literary version of some of the ideas of this paper was posted on the web on the EDGE website at www.edge.org, eliciting close to 600 comments and letters – which helped in the elaboration of this version. The author thanks the commentators and various reviewers, and Yossi Vardi for the material on the events of Sept 18, 2008.

² “Dr. Doom”, *New York Times*, August 15, 2008

³ Note the irony that the ridicule of the warnings in Taleb (2007) and other ideas came from the academic establishment, not from the popular press.

II- INTRODUCTION

Forecasting is a serious professional and scientific endeavor with a certain purpose, namely to provide predictions to be used in formulating decisions, and taking actions. The forecast translates into a decision, and, accordingly, the uncertainty attached to the forecast, i.e., the error, needs to be endogenous to the decision itself. This holds particularly true of risk decisions. In other words, the use of the forecast needs to be determined –or modified – based on the estimated accuracy of the forecast. This, in turn creates an interdependency about what we should or should not forecast –as some forecasts can be harmful to decision makers.

Figure 1 illustrates such example of harm coming from building risk management on the basis of extrapolative (usually highly technical) econometric methods, providing decision-makers with false confidence about the risks, and finding society exposed to several trillions in losses that put capitalism on the verge of collapse.

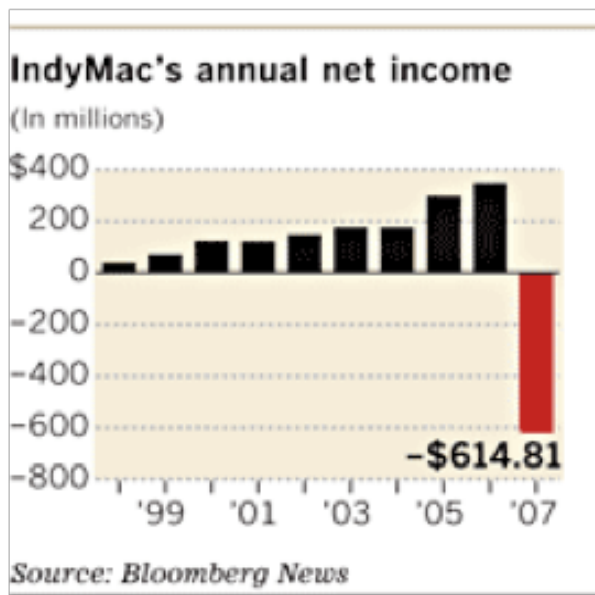


Figure 1- Fat tails at work. Tragic errors from underestimating potential losses, best known cases: FNMA, Freddie Mac, Bear Stearns, Northern Rock, Lehman Brothers, in addition to numerous hedge funds.

A key word here, *fat tails*, implies the outsized role in the total statistical properties coming from one single observation –such as one massive loss coming after years of stable profits or one massive variation unseen in past data.

- “Thin-tails” allow for an ease in forecasting and tractability of the errors;

- “Thick-tails” imply more difficulties in getting a handle on the forecast errors and the fragility of the forecast.

Close to 1000 financial institutions have shut down in 2007 and 2008 from the underestimation of outsized market moves, with losses up to 3.6 trillion⁴. Had their managers been aware of the unreliability of the forecasting methods (which were already apparent in the data), they would have requested a different risk profile, with more robustness in risk management and smaller dependence on complex derivatives.

A- The Smoking Gun

We conducted a simple scientific examination of economic data, using a near-exhaustive set that includes 38 “tradable” variables⁵ that allow for daily prices: major equity indices across the globe (US, Europe, Asia, Latin America), most metals (gold, silver), major interest rate securities, main currencies –what we believe represents around 98% of tradable volume.

We analyzed the properties of the logarithmic returns

$r_{t,\Delta t} = \text{Log}\left(\frac{X_t}{X_{t-\Delta t}}\right)$ where Δt can be 1 day, 10 days, or 66 days (non-overlapping intervals)⁶.

A conventional test of nonnormality used in the literature is the excess kurtosis over the normal distribution. Thus we measured the fourth noncentral

moment $k(\Delta t) = \frac{\sum r_{t,\Delta t}^4}{n}$ of the distributions and focused on the stability of the measurements.

⁴ Bloomberg, Feb 5, 2009.

⁵ We selected a set of near-exhaustive economic data that includes “tradable” securities that allow for a future or a forward market: most equity indices across the globe, most metals, most interest rate securities, most currencies. We collected all available traded futures data –what we believe represents around 98% of tradable volume. The reason we selected tradable data is because of the certainty of the practical aspect of a price on which one can transact: a nontradable currency price can lend itself to all manner of manipulation. More precisely we selected “continuously rolled” futures in which the returns from holding a security are built-in. For instance analyses of Dow Jones that fail to account for dividend payments or analyses of currencies that do not include interest rates provide a bias in the measurement of the mean and higher moments.

⁶ By convention we use $t=1$ as one business day.

Table 1 – Fourth Noncentral Moment at daily, 10-day, 66 day window for the random variables

	K (1)	K(10)	K (66)	Max Quartic	Years
Australian Dollar/USD	6.3	3.8	2.9	0.12	22.
Australia TB 10y	7.5	6.2	3.5	0.08	25.
Australia TB 3y	7.5	5.4	4.2	0.06	21.
BeanOil	5.5	7.	4.9	0.11	47.
Bonds 30Y	5.6	4.7	3.9	0.02	32.
Bovespa	24.9	5.	2.3	0.27	16.
British Pound/USD	6.9	7.4	5.3	0.05	38.
CAC40	6.5	4.7	3.6	0.05	20.
Canadian Dollar	7.4	4.1	3.9	0.06	38.
Cocoa NY	4.9	4.	5.2	0.04	47.
Coffee NY	10.7	5.2	5.3	0.13	37.
Copper	6.4	5.5	4.5	0.05	48.
Corn	9.4	8.	5.	0.18	49.
Crude Oil	29.	4.7	5.1	0.79	26.
CT	7.8	4.8	3.7	0.25	48.
DAX	8.	6.5	3.7	0.2	18.
Euro Bund	4.9	3.2	3.3	0.06	18.
Euro Currency/DEM previously	5.5	3.8	2.8	0.06	38.
Eurodollar Depo 1M	41.5	28.	6.	0.31	19.
Eurodollar Depo 3M	21.1	8.1	7.	0.25	28.
FTSE	15.2	27.4	6.5	0.54	25.
Gold	11.9	14.5	16.6	0.04	35.
Heating Oil	20.	4.1	4.4	0.74	31.
Hogs	4.5	4.6	4.8	0.05	43.
Jakarta Stock Index	40.5	6.2	4.2	0.19	16.
Japanese Gov Bonds	17.2	16.9	4.3	0.48	24.
Live Cattle	4.2	4.9	5.6	0.04	44.
Nasdaq Index	11.4	9.3	5.	0.13	21.
Natural Gas	6.	3.9	3.8	0.06	19.
Nikkei	52.6	4.	2.9	0.72	23.
Notes 5Y	5.1	3.2	2.5	0.06	21.
Russia RTSI	13.3	6.	7.3	0.13	17.
Short Sterling	851.8	93.	3.	0.75	17.
Silver	160.3	22.6	10.2	0.94	46.
Smallcap	6.1	5.7	6.8	0.06	17.
SoyBeans	7.1	8.8	6.7	0.17	47.
SoyMeal	8.9	9.8	8.5	0.09	48.
Sp500	38.2	7.7	5.1	0.79	56.
Sugar #11	9.4	6.4	3.8	0.3	48.
SwissFranc	5.1	3.8	2.6	0.05	38.
TY10Y Notes	5.9	5.5	4.9	0.1	27.
Wheat	5.6	6.	6.9	0.02	49.

Yen/USD 9.7 6.1 2.5 0.27 38.

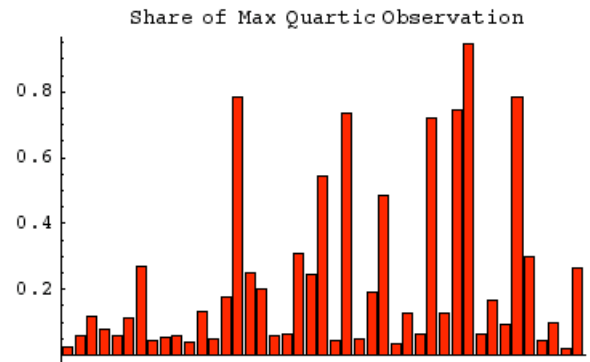


Figure 2 **The Smoking Gun:** Maximum contribution to the Fourth moment Kurtosis coming from the largest observation in ~ 10,000 (29-40 years of daily observations) for 43 economic variables. For the Gaussian the number is expected to be ~.006 for n=10,000.

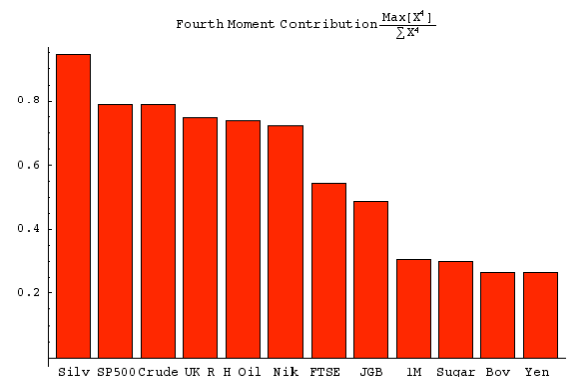


Figure 3- A selection of 12 most acute cases among the 43 economic variables.

By examining Table 1 and Figures 2 and 3, it appears that:

- 1) Economic variables (currency rates, financial assets, interest rates, commodities) are patently fat tailed –to no known exception. The literature (Bundt and Murphy, 2006) shows that this also applies to data not considered here owing to lack of daily changes such as GDP, or inflation.
- 2) Conventional methods, not just those relying on a Gaussian distribution, but those based on least-square methods, or using variance as a measure of dispersion, are according to the data, incapable of tracking the kind of “fat-tails” we see (more technically, in the L^2 norm, as will be discussed in section V). The reason is that most of the kurtosis is concentrated in a few observations, making it

practically unknowable using conventional methods –see Figure 2. Other tests in Section V (the conditional expectation above a threshold) show further instability. This incapacitates least-square methods, linear regression, and similar tools, including risk management methods such as “Gaussian Copulas” that rely on correlations or any form of the product of random variables^{7, 8, 9}.

- 3) There is no evidence of “convergence to Normality” by aggregation, i.e., looking at the kurtosis of weekly or monthly changes. The “fatness” of the tails seems to be conserved under aggregation.

Clearly had decision-makers been aware of such facts, and such unreliability of conventional methods in tracking large deviations, fewer losses would have been incurred as they would have reduced exposures in some areas rather than rely on more “sophisticated” methods. The financial system has been fragile, as this simple test shows, with the evidence staring at us all along.

The Problem of Large Deviations

The empirical problem of small probabilities: The central problem addressed in this paper is that small probabilities are difficult to estimate empirically (since the sample set for these is small), with a greater error rate than the one for more frequent events. But since, in some domains, their effects can be consequential, the error concerning the contribution of small probabilities to the total moments of the distribution becomes disproportionately large. The problem has been dealt with by assuming a probability distribution and extrapolating into the tails –which brings model error into play. Yet, as we will discuss, model error plays a larger role with large deviations.

Links to decision theory: It is not necessary here to argue that a decision maker needs to use a full tableau of payoffs (rather than the simple one-dimensional average forecast) and that payoffs from decisions vary in their sensitivity to forecast errors. For instance, while it is acceptable to take a medicine that might be

effective with a 5% error rate, but offers no side effects otherwise, it is foolish to play Russian roulette with the knowledge one should win with a 5% error rate – indeed standard theory of choice under uncertainty requires the use of full probability distributions, or at least a probability associated with every payoff. But so far this simple truism has not been integrated into the forecasting activity itself –as no classification has been made concerning the tractability and consequences of the errors. To put it simply, the mere separation between forecasting and decisions lacks in both rigor and practicality –as it ruptures the link between forecast error and the quality of the decision.

The extensive literature on decision theory and choices under uncertainty so far has limited itself to 1) assuming *known* probability distributions (except for a few exceptions in which this type of uncertainty has been called “ambiguity”¹⁰), 2) ignoring fat tails. This paper introduces a new structure of fat tails and classification of classes of randomness into the analysis, and focuses on the interrelation between errors and decisions. To establish a link between decision and quality of forecast, this analysis operates along two qualitative lines: qualitative differences between decisions along their vulnerability to error rates on one hand, and qualitative differences between two types of distributions of error rates. So there are two *distinct* types of decisions, and two *distinct* classes of randomness.

This classification allows us to isolate situations in which forecasting needs to be suspended –or a revision of the decision or exposure may be necessary. What we call the “fourth quadrant” is the area in which both the magnitude of forecast errors are large and the sensitivity on these errors is consequential. What we recommend is either changes in the payoff itself (clipping exposure) or the shifting of exposures away from that part. For that we will provide precise rules.

The paper is organized as follows. First, we classify decisions according to targeted payoffs. Second, we discuss the problem of rare events, as these are the ones that are both consequential and hard to predict. Third, we present the classification of the two categories of probability distributions. Finally we present the “fourth quadrant” and what we need to escape it, thus answering the call for how to handle “decision making under low predictability”.

III- THE DIFFERENT TYPES OF DECISIONS

The first type of decisions is simple, it aims at “binary” payoffs, i.e. you just care if something is true or false.

⁷ This should predict, for instance, the total failure in practice of the ARCH/GARCH methods (Engle, 1982), in spite of their successes in-sample, and in academic citations, as they are based on the behavior of squares.

⁸ One counterintuitive result is that sophisticated operators do not seem to be aware of the norm they are using, thus misestimating volatility, see Goldstein and Taleb (2007).

⁹ Practitioners have blamed the naive L-2 reliance in the risk management of credit risk for the blowup of banks in the crisis that started in 2007. See Felix Salmon’s “Recipe For Disaster: The Formula That Killed Wall Street” in Wired. 02/23/2009.

¹⁰ Ellsberg’s paradox, Ellsberg (1961); see also Gardenfors and Sahlin (1982), Levi (1986).

Very true or very false does not matter. Someone is either pregnant or not pregnant. A biological experiment in the laboratory or a bet about the outcome of an election belong to this category. A scientific statement is traditionally considered "true" or "false" with some confidence interval. More technically, they depend on the zeroth moment, namely just on probability of events, and not their magnitude—for these one just cares about "raw" probability¹¹.

Clearly these are not very prevalent in life—they mostly exist in laboratory experiments and in research papers.

The second type of decisions depends on more complex payoffs. The decision maker does not just care of the frequency—but of the impact as well, or, even more complex, some function of the impact. So there is another layer of uncertainty of impact. These depend on higher moments of the distribution. When one invests one does not care about the frequency, how many times he makes or loses, he cares about the expectation: how many times money is made or lost *times* the amount made or lost. We will see that there are even more complex decisions.

More formally, where $p[x]$ is the probability distribution of the random variable x , \mathcal{D} the domain on which the distribution is defined, the payoff $\lambda(x)$ is defined by integrating on \mathcal{D} as:

$$\lambda(x) = \int f(x) p(x) dx$$

Note that we can incorporate utility or nonlinearities of the payoff in the function $f(x)$. But let us ignore utility for the sake of simplification.

For a simple payoff, $f(x) = 1$. So $L(x)$ becomes the simple probability of exceeding x , since the final outcome is either 1 or 0 (or 1 and -1).

For more complicated payoffs, $f(x)$ can be complex. If the payoff depends on a simple expectation, i.e., $\lambda(x) = E[x]$, the corresponding function $f(x)=x$, and we need to ignore frequencies since it is the payoff that matters. One can be right 99% of the time, but it does not matter at all since with some skewed distribution, the consequence on the expectation of the 1% error can be too large. Forecasting typically has $f(x)=x$, a linear function of x , while measures such as least squares depend on the higher moments $f(x)=x^2$.

Note that some financial products can even depend on the fourth moment¹².

Table 2 Tableau of Decisions

Mo "True/False" $f(x)=0$	M1 Expectations LINEAR PAYOFF $f(x)=1$	M2+ NONLINEAR PAYOFF $f(x)$ nonlinear(= x^2, x^3 , etc.)
Medicine (health not epidemics)	Finance : nonleveraged Investment	Derivative payoffs
Psychology experiments	Insurance, measures of expected shortfall	Dynamically hedged portfolios
Bets (prediction markets)	General risk management	Leveraged portfolios (around the loss point)
Binary/Digital derivatives	Climate	Cubic payoffs (strips of out of the money options)
Life/Death	Economics (Policy)	Errors in analyses of volatility
	Security: Terrorism, Natural catastrophes	Calibration of nonlinear models
	Epidemics	Expectation weighted by nonlinear

¹¹ The difference can be best illustrated as follows. One of the most erroneous comparisons encountered in economics is the one between "wine rating" and "credit rating" of complex securities. Errors in wine rating are hardly consequential for the buyer (the "payoff" is binary); errors in credit ratings bankrupted banks as these carry massive payoffs.

¹² More formally, a linear function with respect to the variable x has no second derivative; a convex function is one with a positive second derivative. By expanding the expectation of $f(x)$ we end up with $E[f(x)] = f(x) e[\Delta x] + \frac{1}{2} f''(x) E[\Delta x^2] + \dots$ hence higher orders matter to the extent of the importance of higher derivatives.

		utility
	Casinos	Kurtosis-based positioning ("volatility trading")

Next we turn to a discussion of the problem of rare events.

IV- THE PROBLEM OF RARE EVENTS

The passage from theory to the real world presents two distinct difficulties: "inverse problems" and "pre-asymptotics".

Inverse Problems. It is the greatest difficulty one can encounter in deriving properties. In real life we do not observe probability distributions. We just observe events. So we do not know the statistical properties—until, of course, after the fact—as we can see in Figure 1. Given a set of observations, plenty of statistical distributions can correspond to the exact same realizations—each would extrapolate differently outside the set of events on which it was derived. The inverse problem is more acute when more theories, more distributions can fit a set a data—particularly in the presence of nonlinearities or nonparsimonious distributions¹³.

So this inverse problem is compounded two problems:

+ *The small sample properties of rare events* as these will be naturally rare in a past sample. It is also acute in the presence of nonlinearities as the families of possible models/parametrization explode in numbers.

+ *The survivorship bias effect of high impact rare events.* For negatively skewed distributions (with a thicker left tail), the problem is worse. Clearly, catastrophic events will be necessarily absent from the data—since the survivorship of the variable itself will depend on such effect. Thus left tailed distributions will 1) overestimate the mean; 2) underestimate the variance and the risk.

Figure 4 shows how we normally lack data in the tails; Figure 5 shows the empirical effect.

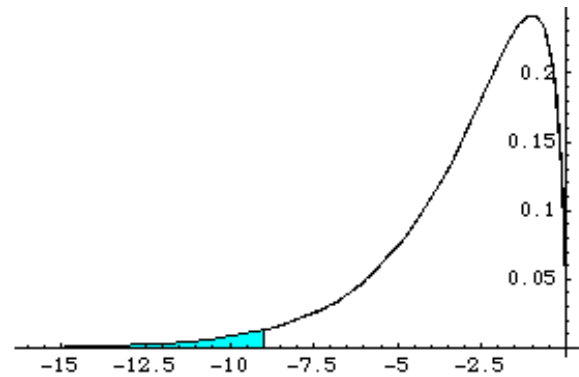
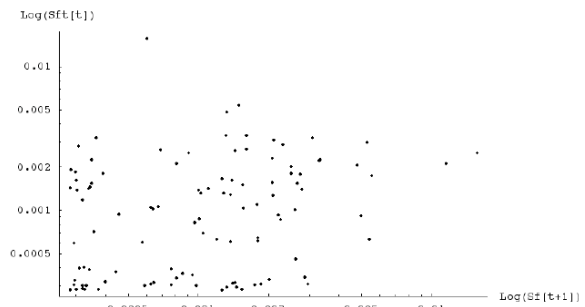


Figure 4 The Confirmation Bias At Work. The shaded area shows what tend to be missing from the observations. For negatively-skewed, fat-tailed distributions, we do not see much of negative outcomes for surviving entities AND we have a small sample in the left tail. This illustrates why we tend to see a better past for a certain class of time series than warranted.



*Figure 5 **Outliers don't Predict Outliers.** The plot shows (in Logarithmic scale) a shortfall in one given year against the shortfall the following one, repeated throughout for the 43 variables. A shortfall here is defined as the sum of deviations in excess of 7%. Past large deviations do not appear to predict future large deviations, at different lags.*

¹³ A Gaussian distribution is parsimonious (with only two parameters to fit). But the problem of adding layers of possible jumps, each with a different probabilities opens up endless possibilities of combinations of parameters.

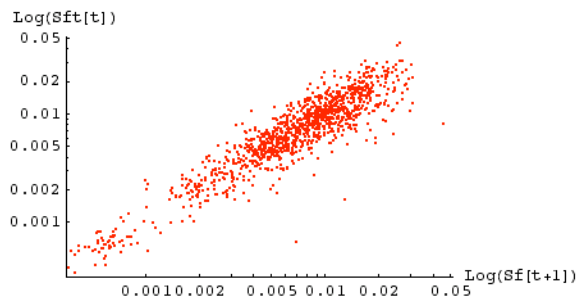


Figure 6 Regular Events Predict Regular Events. This plot shows, by comparison with Figure 5, how, for the same variables, mean deviation in one period predicts the one in the subsequent period.

Pre-asymptotics. Theories can be extremely dangerous when they were derived in idealized situations, the asymptote, but are used outside the asymptote (its limit, say infinity or the infinitesimal). Some asymptotic properties do work well preasymptotically (as we'll see, with type-1 distributions), which is why casinos do well, but others do not, particularly when it comes to the class of fat-tailed distributions.

Most statistical education is based on these asymptotic, laboratory-style Platonic properties—yet we take economic decisions in the real world that very rarely resembles the asymptote. Most of what students of statistics do is assume a structure, typically with a known probability. Yet the problem we have is not so much making computations once you know the probabilities, but finding the true distribution.

V- THE TWO PROBABILISTIC STRUCTURES

There are two classes of probability domains—very distinct qualitatively and quantitatively—according to precise mathematical properties. The first, Type-1, we call “benign” thin-tailed nonscalable, the second, Type 2, “wild” thick tailed scalable, or fractal (the attribution “wild” comes from Mandelbrot’s classification of Mandelbrot[1963]).

Taleb (2009) shows that one of the mistakes in the economics literature that “fattens the tails”, with two main classes of nonparsimonious models and processes (the jump-diffusion processes of Merton, 1973¹⁴ or stochastic volatility models such as Engels’ ARCH¹⁵) is to believe that the second type of distributions are

¹⁴ See the general decomposition into diffusion and jump (non-scalable) in Merton(1976), Duffie, Pan, and Singleton(2000); discussion in Baz and Chacko (2004), Haug (2007).

¹⁵ Engle(1982).

amenable to analyses like the first—except with fatter tails. In reality, a fact commonly encountered by practitioners, fat-tailed distributions are very unwieldy—as we can see in Figure 2. Furthermore we often face a problem of mistaking one for the other: a process that is extremely well behaved, but, on the occasion, delivers a very large deviation, can be easily mistaken for a thin-tailed one—a problem known as the “problem of confirmation” (Taleb, 2007). So we need to be suspicious of the mistake of taking Type-2 for Type-1 as it is more severe (and more readily made) than the one in the other direction¹⁶.

As we will saw from the data presented, this classification, “fat tails” does not just mean having a fourth moment worse than the Gaussian. The Poisson distribution, or a mixed distribution with a known Poisson jump, would have tails thicker than the Gaussian; but this mild form of fat tails can be dealt with rather easily—the distribution has all its moments finite. The problem comes from the structure of the decline in probabilities for larger deviations and the ease with which the tools at our disposal can be tripped into producing erroneous results from observations of data in a finite sample and jumping to wrong decisions.

The scalable property of Type-2 distributions:

Take a random variable x . With scalable distributions, asymptotically, for x large enough, (i.e. “in the tails”),

$$\frac{P[X > n x]}{P[X > x]}$$

depends on n , not on x (the same

property can hold for $P[X < n x]$ for negative values). This induces statistical self-similarities. Note that owing to the finiteness of the realizations of random variables, and lack of samples in the tails we might not be able to observe such property—yet not be able to rule out.

For economic variables, there is no fundamental reason for the ratio of “exceedances” (i.e., the cumulative probability of exceeding a certain threshold) to decline as both the numerator and the denominators are multiplied by 2.

This self-similarity at all scales generates power-law, or Paretian, tails, i.e., above a crossover point, $P[X > x] = K x^{-\alpha}$.^{17 18}

¹⁶ Makridakis et al(1993), Makridakis and Hibon (2000) present evidence that more complicated methods of forecasting do not deliver superior results to simple ones (already bad). The obvious reason is that the errors in calibration swell with the complexity of the model.

¹⁷ Scalable discussions: introduced in Mandelbrot(1963), Mandelbrot (1997), Mandelbrot and Taleb (2009).

¹⁸ Complexity and power laws: Sornette (2004), Stanley et al (2000), Amaral et al (1997); for scalability in different aspects of financial data, Gabaix et al. (2003a,2003b,2003c), Gopikrishnan et al. (1998,1999,2000), Plerou et al (2000). For the statistical mechanics of scale-free networks Barabasi and Albert (1999), Albert and Barabasi(2000),Albert et Al (2002). The “sandpile effect” (i.e., avalanches and cascades) is

Let's draw the implications of type-2 distributions:

Finiteness of Moments and Higher Order Effects.

For thick tailed distributions, moments higher than α are not "finite", i.e., they cannot be computed. They can certainly be measured in finite samples –thus giving the illusion of finiteness. But they typically show a great degree of instability. For instance distribution with infinite variance will always provide, in a sample, the illusion of finiteness of variance.

In other words, while for type-1 errors converge (the expectations of higher orders of x , say of order n , such as $1/n! E[x^n]$, where x is the error, do not decline. In fact they become explosive).

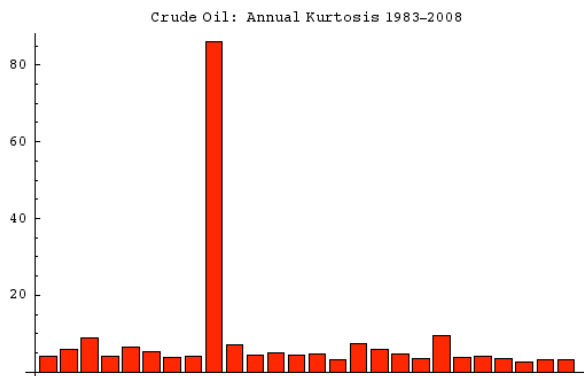


Figure 7-Kurtosis over time: example of an "infinite moment". The graph shows the fourth moment for crude oil in annual nonoverlapping observations between 1982 and 2008. The instability show in the dependence of the measurement on the observation window.

2) "Atypicality" of Moves. For thin tailed domains, the conditional expectation of a random variable X , conditional on its exceeding a number K , converge to K for larger values of K .

$$\lim_{K \rightarrow \infty} E[X |_{X > K}] = K$$

For instance the conditional expectation for a Gaussian variable (assuming a mean of 0) conditional that the variable exceeds 0 is approximately .8 standard deviations. But with K equals 6 standard deviations, the conditional expectation converges to 6 standard deviations. The same applies to all the random variables that do not have a Paretan tail. This induces some "typicality" of large moves.

For fat tailed variables, such limit does not seem to hold:

discussed in Bak et al (1987, 1988), Bak (1996), as power laws arise from conditions of self-organized criticality.

$$\lim_{K \rightarrow \infty} E[X |_{X > K}] = K c$$

where c is a constant. For instance, the conditional expectation of a market move, given that it is in excess of 3 mean deviations, will be around 5 mean deviations. The expectation of a move conditional on it being higher than 10 mean deviations will be around 18. This property is quite crucial.

The atypicality of moves has the following significance.

- One may correctly predict a given event, say, a war, a market crash, a credit crisis. But the amplitude of the damage will be unpredicted. The open-endedness of the outcomes can cause severe miscalculation of the expected payoff function. For instance the investment bank Morgan Stanley predicted a credit crisis but was severely hurt (and needed to be rescued) because it did not anticipate the extent of the damage.
- Methods like Value-at-Risk¹⁹ that may correctly compute, say, a 99% probability of not losing no more than a given sum, called "value-at-risk", will nevertheless miscompute the conditional expectation should such threshold be exceeded. For instance one has 99% probability of not exceeding a \$1 million loss, but should such a loss occur, it can be \$10 million or \$100 million.

This lack of typicality is of some significance. Stress testing and scenario generation are based on assuming a "crisis" scenario and checking robustness to it. Unfortunately such luxury is not available for fat tails as "crisis" does not have a typical magnitude.

The following table shows the evidence of lack of convergence to thin tails –hence lack of "typicality" of the moves. We stopped for segments for which the number of observations becomes small –since lack of observations in the tails can provide the illusion of "thin" tails.

Table 3- Conditional expectation for moves $> K$, 43 economic variables

K Mean Deviations	Mean Move (in MAD) in excess of K	n
1	2.01443	65958
2	3.0814	23450
3	4.19842	8355

¹⁹ For the definition of Value at Risk, Jorion (2001); critique: Joe Nocera, "Risk Mismanagement: What led to the Financial Meltdown", *New York Time Magazine*, Jan 2, 2009

4	5.33587	3202
5	6.52524	1360
6	7.74405	660
7	9.10917	340
8	10.3649	192
9	11.6737	120
10	13.8726	84
11	15.3832	65
12	19.3987	47
13	21.0189	36
14	21.7426	29
15	24.1414	21
16	25.1188	18
17	27.8408	13
18	31.2309	11
19	35.6161	7
20	35.9036	6

Table 4 Conditional expectation for moves < K, 43 economic variables

K	Average Move (in MAD) below K	n
Mean Deviations		
-1	-2.06689	62803
-2	-3.13423	23258
-3	-4.24303	8676
-4	-5.40792	3346
-5	-6.66288	1415
-6	-7.95766	689
-7	-9.43672	392
-8	-11.0048	226
-9	-13.158	133
-10	-14.6851	95
-11	-17.02	66
-12	-19.5828	46
-13	-21.353	38
-14	-25.0956	27
-15	-25.7004	22
-16	-27.5269	20
-17	-33.6529	16
-18	-35.0807	14
-19	-35.5523	13
-20	-38.7657	11

3) **Preasymptotics:** Even if we eventually converge to a probability distribution, of the kind well known and tractable, it is central that time to convergence plays a large role.

For instance, much of the literature invokes the Central Limit Theorem to assume that fat-tailed distribution with finite variance converge to a Gaussian under summation. If daily errors are fat-tailed, cumulative monthly errors will become Gaussian. In practice, this does not appear to hold. The data in the appendix show that economic variables do not remotely converge to the Gaussian under aggregation.

Furthermore, finiteness of variance is necessary but highly insufficient a condition. Bouchaud and Potters

[2003] showed that the tails remain heavy while the body of the distribution becomes Gaussian.

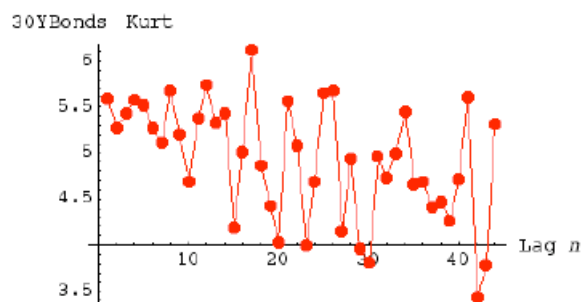


Figure 8- Behavior of Kurtosis under aggregation: we lengthen the window of changes from 1 day to 50 days. Even for variables with infinite fourth moment, the kurtosis tends to drop under aggregation in small samples, then rise abruptly after a large observation.

4) Metrics.

Much of times series work seems to be based on metrics in the square domain, hence patently intractable. Define the norm L^p :

$$\left(\frac{1}{n} \sum |x|^p\right)^{1/p}$$

it will increase along with p . The numbers can become explosive, with rare events taking a disproportionately larger share of the metric at higher orders of p . Thus variance/standard deviation ($p=2$), as a measure of dispersion, will be far more unstable than mean deviation ($p=1$). The ratio of mean-deviation to variance (Taleb, 2009) is highly unstable for economic variables. Thus modelizations based on variance become incapacitated. More practically, this means that for distribution with finite mean (tail exponent greater than 1), the mean deviation is more "robust"²⁰.

5) Incidence of Rare Events

²⁰ Note on the weaknesses of nonparametric statistics: Mean deviation is often used as robust, nonparametric or distribution-free statistic. It does work better than variance, as we saw, but does not contain information on rare events by the argument seen before. Likewise nonparametric statistical methods (relying on empirical frequency) will be extremely fragile to the "black swan problem", since the absence of large deviations in the past leave us in a near-total opacity about their occurrence in the future –as we saw in Figure 4, these are confirmatory. In other words nonparametric statistics, those that consist in fitting a kernel to empirical frequencies, assume, even more than other methods, that a large deviation will have a predecessor.

One common error is to believe that thickening the tails leads to an *increase* of the probability of rare events. In fact, it usually leads to the decrease of incidence of such events, but the magnitude of the event, should it happen, will be much larger.

Take, for instance, a normally distributed random variable. The probability of exceeding 1 standard deviation is about 16%. Observed returns in the markets, with a higher kurtosis, present a lower probability, around 7-10% of exceeding the same threshold—but the depth of the excursions is greater.

6) Calibration Errors and Fat Tails

One does not need to accept power laws to use them. A convincing argument is that if we don't know what a "typical" event is, fractal power laws are the most effective way to *discuss* the extremes mathematically. It does not mean that the real world generator is actually a power law—it means that we don't understand the structure of the external events it delivers and need a tool of analysis so you do not become a turkey. Also, fractals simplify the mathematical discussions because all you need is perturbate one parameter, here the α , and it increases or decreases the role of the rare event in the total properties.

Say, for instance, that, in an analysis, you move α from 2.3 to 2 for data in the publishing business; the sales of books in excess of 1 million copies would triple! This method is akin to generating combinations of scenarios with series of probabilities and series of payoffs, fattening the tail at each time.

The following argument will help illustrate the general problem with forecasting under fat tails. Now the problem: *Parametrizing a power law lends itself to extremely large estimation errors* (since heavy tails have inverse problems). Small changes in the α main parameter used by power laws leads to extremely large effects in the tails. Monstrous.

And we don't observe the α --an uncertainty that comes from the measurement error. Figure 9 shows more than 40 thousand computations of the tail exponent α from different samples of different economic variables (data for which it is impossible to refute fractal power laws). We clearly have problems figuring it what the α is: our results are marred with errors. The mean absolute error in the measurement of the tail exponent is in excess of 1 (i.e. between $\alpha=2$ and $\alpha=3$). Numerous papers in econophysics found an "average" alpha between 2 and 3—but if you process the >20 million pieces of data analyzed in the literature,

you find that the variations between single variables are extremely significant²¹.

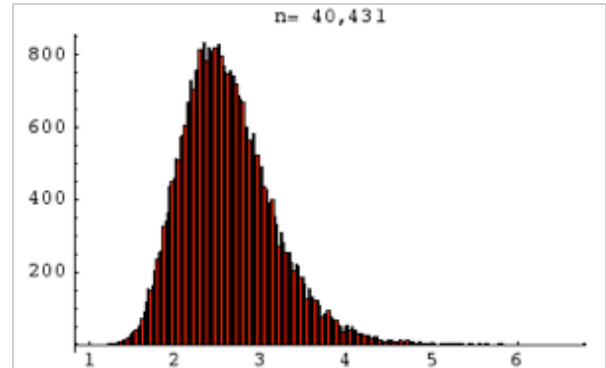


Figure 9 Estimation error in α from 40 thousand economic variables.

Now this mean error has massive consequences. Figure 10 shows the effect: the expected value of your losses in excess of a certain amount (called "shortfall") is multiplied by >10 from a small change in the α that is less than its mean error²².

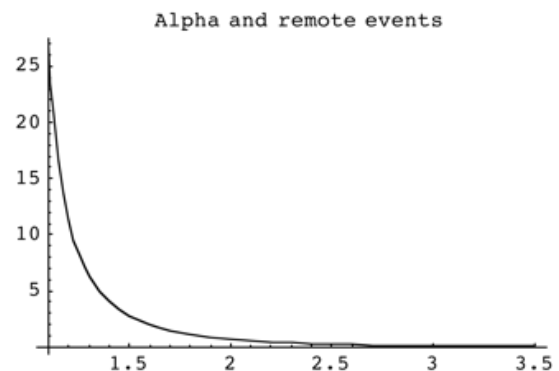


Figure 10 The value of the expected shortfall (expected losses in excess of a certain threshold) in response to changes in tail exponent α . We can see it explode by an order of magnitude.

²¹ One aspect of this inverse problem is even pervasive in Monte Carlo experiments (much better behaved than the real world), see Weron (2001).

²² Note that the literature on extreme value theory (Embrecht et al., 1997) does not solve much of the problem as the calibration errors stay the same. The argument about calibration we saw earlier makes the values depend on the unknowable tail exponent. This calibration problem explains how Extreme Value Theory works better on computers than in the real world (and has failed completely in the economic crisis of 2008-2009).

V- THE MAP

First Quadrant: Simple binary decisions, under type-1 distributions: forecasting is safe. These situations are, unfortunately, more common in laboratories and games than real life. We rarely observe these in payoffs in economic decision making. Examples: some medical decisions, casino bets, prediction markets.

Second Quadrant: Complex decisions under type-1 distributions: Statistical methods may work satisfactorily, though there are some risks. True, thin-tails may not be a panacea owing to preasymptotics, lack of independence, and model error. There, clearly, are problems there, but these have been addressed extensively in the literature (see Freedman, 2007).

Third Quadrant: Simple decisions, under type-2 distributions: there is little harm in being wrong –the tails do not impact the payoffs.

Fourth Quadrant: Complex decisions under type-2 distributions: that is where the problem resides. We need to avoid prediction of remote payoffs—though not necessarily ordinary ones. Payoffs from remote parts of the distribution are more difficult to predict than closer parts.

A general principle is that, while in the first three quadrants you can use *the best* model you can find, this is dangerous in the fourth quadrant: no model should be better than just any model. So the idea is to exit the fourth quadrant.

The recommendation is to move into the third quadrant –it is not possible to change the distribution; it is possible to change the payoff , as will be discussed in the next section.

Distribution 2 (no or unknown characteristic scale)	Third Quadrant: Safe	Fourth Quadrant: Dangers²³
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Table 5 The Four Quadrants.

	Simple payoffs	Complex payoffs
Distribution 1 (“thin tailed”)	First Quadrant Extremely Safe	Second Quadrant: Safe

²³ The dangers are limited to exposures in the negative domain (i.e., adverse payoffs). Some exposures, we will see, can be only “positive”.

The subtlety is that, while we have a poor idea about the expectation in the 4th quadrant, exposures to rare events are not symmetric.

VI- DECISION-MAKING AND FORECASTING IN THE FOURTH QUADRANT

1- Solutions by changing the payoff:

Finally, the main idea proposed in this paper is to endogenize decisions, i.e., escape the 4th quadrant whenever possible by changing the payoff in reaction to the high degree of unpredictability and the harm it causes. How?

Just consider that the property of “atypicality” of the moves can be compensated by truncating the payoffs, thus creating an organic “worst case” scenario that is resistant to forecast errors. Recall that a binary payoff is insensitive to fat tails precisely because above a certain level, the domain of integration, changes in probabilities do not impact the payoff. So making the payoff no longer open-ended mitigates the problems, thus making it more tractable mathematically.

A way to express it using moments: all moments of the distribution become finite in the absence of open-ended payoffs –by putting a floor L below which $f(x) = 0$, as well a ceiling H . Just consider that if you are integrating payoffs in a finite, not open-ended domain, i.e. between L and H , respectively, *the tails of the distributions outside that domain no longer matter*. Thus the domain of integration becomes the domain of payoff.

$$\lambda(x) = \int_L^H f(x) p(x) dx$$

With an investment portfolio, for instance, it is possible to “put a floor” on the payoff using insurance, or, better even, by changing the allocation. Insurance products are tailored with a maximum payoff; catastrophe insurance products are also set with a “cap”, though the cap might be high enough to allow for a dependence on the error of the distribution²⁴.

The Effect of Skewness: We omitted earlier to discuss asymmetry in either the payoff or in the distribution. Clearly the Fourth Quadrant can present left or right skewness. If we suspect right-skewness, the true mean is more likely to be underestimated by measurement of past realizations, and the total potential is likewise poorly gauged. A biotech company

²⁴ Insurance companies might cap the payoff of a single claim, but a collection of capped claims might represent some problems as the maximum loss becomes too large as to be almost undistinguishable from that with an uncapped payoff.

(usually) faces positive uncertainty, a bank faces almost exclusively negative shocks.

More significantly, by raising the L (the lower bound), one can easily produce positive skewness, with a set floor for potential adverse outcomes and open upside. For instance what Taleb calls a “barbell” investment strategy consists in allocating a high portion of a portfolio to T-Bills (or equivalent), say α , with $0 < \alpha < 1$, and a small portion $(1-\alpha)$ to high-variance securities. While the total portfolio has medium variance, $L = (1-\alpha)$ times the face value invested while another portfolio of the same variance might lose 100%.

Convex and Concave to Error: More generally, we can consider concave to model error if the payoff from the error (obtained by changing the tails of the distribution) has a negative second derivative with respect to that change in the tails, or is negatively skewed (like the payoff of a short option). It will be convex if the payoff is positively skewed, (like the payoff of a long option).

The Effect of Leverage in Operations and Investment

Leveraging in finance has the effect of increasing concavity to model error. As we will see, it is exactly the opposite of redundancy –it causes payoffs to increase, but at the costs of an absorbing barrier should there be an extreme event that exceeds the allowance made in the risk measurement. Redundancy, on the other hand, is the equivalent of de-leveraging, i.e. by having more idle “inefficient” capital on the side. But a second look at such funds can reveal that there may be a direct expected value from being able to benefit from opportunities in the event of asset deflation – hence “idle” capital needs to be analyzed as an option.

2- Solutions by mitigating forecasting errors

Optimization v/s Redundancy. The optimization paradigm of the economics literature meets some problems in the fourth quadrant: what if we have a consequential forecasting error? Aside from the issue that the economic agent is optimizing on the future states of the world, with a given probability distribution, nowhere²⁵ have the equations taken into account the possibility of a large deviation that would allow *not* optimizing consumption and having idle capital. Also, the psychological literature on well-being (Kahneman, 1999) shows an extremely concave utility function of income –if one spends such income. But if one hides it under the mattress, he will be less vulnerable to an

²⁵ Merton 1992, for a discussion of the general consumption Capital Asset Pricing Market.

extreme event. So there is an enhanced survival probability for those who have additional margin.

While economics have been mired in conventional linear analysis, stochastic optimization with Bellman-style equations that fall into the category Type-1, some intuitions of the point are provided by complex systems. One of the central attributes of complex systems is redundancy (May et al, 2008).

Biological systems—those that survived millions of years—include a large share of redundancies^{26 27}. Just consider the number of double organs (lungs, kidneys, ears). This may suggest an option-theoretic analysis: redundancy is like an option. One certainly pay for it, but it may be necessary for survival. And while redundancy means similar functions used by identical organs or resources, biological systems have, in addition, recourse to “degeneracy”, the possibility of one organ to perform more than one function, which is the analog of redundancy at a functional level (Edelman and Gally, 2001).

When institutions such as banks optimize, they often do not realizing that a simple model error can blow through their capital (as it just did).

Type-1 Noise



Type-2 Noise

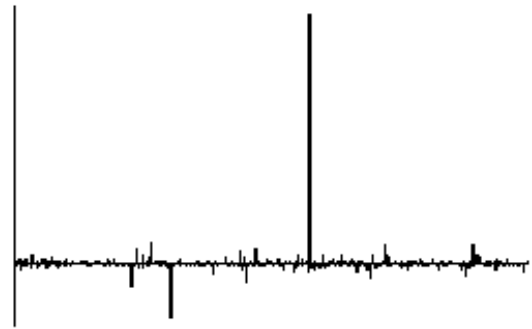


Figure 11- Comparison between Gaussian-style noise and Type-2 noise with extreme spikes – which necessitates more redundancy (or insurance) than required. Policymakers and forecasters were not aware that complex systems tend to produce the second type of noise.

Examples: In one day in August 2007, Goldman Sachs experienced 24 time the average daily transaction volume²⁸—would 29 times have blown up the clearing system? Another severe instance of an extreme “spike” lies in an event of September 18, 2008, in the aftermath of the Lehman Brothers Bankruptcy. According to congress documents, only made public in February 2009.

On Thursday (Sept 18), at 11am the Federal Reserve noticed a tremendous draw-down of money market accounts in the U.S., to the tune of \$550 billion was being drawn out in the matter of an hour or two.

If they had not done that[add liquidity], their estimation is that by 2pm that afternoon, \$5.5 trillion would have been drawn out of the money market system of the U.S., would have collapsed the entire economy of the U.S., and within 24 hours the world economy would have collapsed. It would have been the end of our economic system and our political system as we know it²⁹.

For naive economics, the best way to effectively reduce costs is to minimize redundancy, hence avoiding the option premium of insurance. Indeed some systems tend to optimize—therefore become more fragile. Barabasi and Albert (1999), Albert and Barabasi (2002) warned (ahead of the North Eastern power outage of August 2003) how electricity grids for example optimize to the point of not coping with unexpected surges – which predicted the possibility of a blackout of the magnitude of the one that took place in the North Eastern U.S. in August 2003. We cannot discuss “flat

²⁶ May et al. (2008)

²⁷ For the scalability of biological systems, see Burlando (1993), Harte et al. (1999), Solé et al (1999), Ritchie et al (1999), Enquist and Niklas (2001).

²⁸ Personal communication, Pentagon Highland Forum, April meeting, 2008.

²⁹ http://www.liveleak.com/view?i=ca2_1234032281

earth" globalization without realizing that it is overoptimized to the point of maximal vulnerability.

2-b Time. It takes much, much longer for a fat-tailed time series to reveal its properties –in fact many can in short episodes masquerade as thin-tailed. At the worst, we don't know how long it would take to know. But we can have a pretty clear idea if *organically*, because of the nature of the payoff, the "Black Swan" can hit on the left (losses) or on the right (profits). The point can be used in climatic analysis. Things that have worked for a long time are preferable—they are more likely to have reached their ergodic states.

2-c The Problem of Moral Hazard. Is optimal to make series of annual bonuses betting on hidden risks in the Fourth Quadrant, then "blow up" (Taleb, 2004). The problem is that bonus payments are made with a higher frequency (i.e. annual) than warranted from the statistical properties (when it takes longer to capture the statistical properties).

2-d Metrics. Conventional metrics based on type 1 randomness fail to produce reliable results –while the economics literature is grounded in them. Concepts like "standard deviation" are not stable and do not measure anything in the Fourth Quadrant. So does "linear regression" (the errors are in the fourth quadrant), "Sharpe ratio", Markowitz optimal portfolio³⁰, ANOVA, Least square, etc. "Variance"/"standard deviation" are terms invented years ago when we had no computers. Note that from the data shown and the instability of the kurtosis, no sample will ever deliver the true variance in reasonable time. Yet, note that truncating payoffs blunt the effects of the inadequacy of the metrics.

VI- CONCLUSION

To conclude, we offered a method of robustifying payoffs from large deviations and making forecasts possible to perform. The extensions can be generalized to larger notion of society's safety – for instance how we should build systems (internet, banking structure, etc.) impervious to random effects.

³⁰ The framework of Markowitz (1952) as it is built on L^2 norm, does not stand any form of empirical or even theoretical validity, owing to the dominance higher moment effects, even in the presence of "finite" variance, see Taleb (2009).

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