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The Principle of Minimum Differentiation Reconsidered: Some New Developments in the Theory of Spatial Competition \(^1,^2\)

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I. INTRODUCTION

In his famous 1929 paper, \([14]\), Hotelling presented a model of two firms competing to sell a homogeneous product to customers spread evenly along a linear market. In equilibrium the duopolists are located at the centre of the market rather than being in the locations that would minimize transport costs. Hotelling originally suggested that his model explained a wide variety of social phenomena.

Boulding, who appears to have been the originator of the term *principle of minimum differentiation* (called *MD* hereafter) to describe Hotelling's result, is even more extravagant in his suggestions as to the range of phenomena that are explained by the model. He writes \([5, p. 484]\):

This is a principle of the utmost generality. It explains why all the dime stores are usually clustered together, often next door to each other; why certain towns attract large numbers of firms of one kind; why an industry, such as the garment industry, will concentrate in one quarter of a city. It is a principle which can be carried over into other "differences" than spatial differences. The general rule for any new manufacturer coming into an industry is "make your product as like the existing products as you can without destroying the differences". It explains why all automobiles are so much alike and why no manufacturer dares make a car in which a tall hat can be worn comfortably. It even explains why Methodists, Baptists, and even Quakers are so much alike, and tend to get even more alike.

Hotelling's model has been criticized and extended in the 40-odd years since its publication. It has also been applied to a number of specific cases (see e.g. \([25]\)). In the present paper we have set ourselves the tasks of examining, in a more systematic fashion than has been done to date, the cases to which *MD* applies and of discovering other principles applicable to small-group competition. We consider how robust is the tendency

\(^1\) *First version received February 1972; final version accepted December 1973 (Eds.).*  
\(^2\) We are indebted to many colleagues for comments and suggestions. We are also indebted to the Queen's Institute for Economic Research for generous support over a two year period.
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toward MD in the face of changes in the specification of the model. Five assumptions seem critical: (a) the nature of the consumers' demand (either one unit per period at a parametric price or completely inelastic); (b) the number of firms is restricted to two; (c) each firm adopts zero conjectural variation (called ZCV hereafter) with respect to the behaviour of the other firm; (d) the firms compete in a linear market that has boundaries; and (e) the customers are evenly spread throughout the market. We do not study variations in assumption (a) because they have been dealt with extensively elsewhere.

Although the analysis is in terms of location theory, many of the results generalize to other forms of differentiation, and the conditions under which they do are mentioned briefly in footnote 3, p. 46.

The paper is divided into two parts: one-dimensional markets and two-dimensional markets. Within each we distinguish (a) bounded, (b) unbounded but finite, and (c) unbounded, infinite spaces. Among other things, we show: in one dimension the nature of the space is not, as many investigators have thought, critical; in two dimensions, however, the very existence of equilibrium may depend upon the nature of the space; the commonly-used rectangular customer density function 1 yields results that do not generalize to any other density function; multiple equilibria occur in many simple models; and MD occurs only when the number of firms is restricted to two.

The assumptions of our basic model are as follows:

(i) Customers are distributed throughout the market according to a customer density function which is integrable and once differentiable.
(ii) Each customer purchases one unit of the product per unit of time.
(iii) Transport costs are an increasing function of distance.
(iv) Customers always buy from the firm that quotes the lowest delivered price (mill price plus transport cost).
(v) All firms charge the same parametric mill price.
(vi) Production is at constant marginal cost which is less than the mill price.
(vii) There are no costs of relocation. Presented with the chance of changing its location to change its market from \( M_1 \) to \( M_2 \), the firm will move if \( M_2 \) is preferred to \( M_1 \); it will remain where it is if \( M_1 \) is the same as or preferred to \( M_2 \).
(viii) No more than one firm can occupy a given location.
(ix) In choosing its location, the \( i \)th firm conjectures either (a) that all other firms will leave their own location unaltered (ZCV) 2; or (b) that some other firm, \( j \), will change its location in a way that causes the maximum possible loss of market to \( i \).
(x) Firms seek to maximize their profits subject to their conjectural variation.

In Section IV we discuss the consequences of altering assumptions (ii), (iv), (v) and (vi).

A few terms that are used throughout the paper need to be defined.

The \( i \)th Firm's Market Boundary. (a) An interior boundary is the locus of points that are equidistant from the \( i \)th firm and one other firm, and not closer either to any other firm or to some portion of the market boundary; (b) an exterior boundary is that portion of the market boundary that is closer to the \( i \)th firm than to any other firm.

An Interior Firm. A firm whose entire market boundary is an interior boundary.

A Peripheral Firm. A firm whose market boundary is an exterior boundary over some of its range.

1 The density function most commonly encountered in the literature is rectangular (see e.g. [3] [6], [9], [14], [17], [18], [24], [27] and [28]).

2 Although not applicable in all situations, ZCV is a reasonable assumption, either where the equilibrium is approached very rapidly so that firms do not have time to learn their opponents' reactions, or where relocation occurs with a long time lag (because, e.g., relocation is very costly) as with many locational problems.
Paired Firms. Two firms are said to be paired when the distance between them is as small as is permitted. The minimum permitted distance, $\delta$, is arbitrary and its size is unimportant as long as it is "small" in relation to the overall size of the market.

Minimum Differentiation. This is said to occur when all firms in the market are separated from their neighbours by the distance $\delta$.

Equilibrium. The $i$th firm is in equilibrium when there is no location that is preferred to its present location. The whole market is in equilibrium whenever all $n$ firms are individually in equilibrium.

II. ONE-DIMENSIONAL MARKETS

In one-dimensional markets the location of the $i$th firm divides its market segment into two sides. Where the two sides are unequal they are referred to as the long and short sides of the firm's market. Each side is also referred to as a half-market (whether or not the sides are of equal length).

Assumption (iv) implies that if the $i$th firm is an interior firm, its market extends half the distance to its two neighbours. The length of an interior firm's market is thus half the length of the interval between its two neighbours wherever the firm locates within that interval. If it is a peripheral firm, its market extends all the way to the market boundary in one direction and half way to its one neighbour in the other direction.

When two firms are paired, the short side of each of their markets is $\delta/2$. It is assumed for ease of analysis that the short side of the market is zero for paired firms (actually it goes to zero as $\delta \to 0$).

![Figure 1](image_url)

Firms are indicated by numbers and the boundaries of the firms' markets by broken lines.

Model 1. The assumptions that distinguish this model are $ZCV$ and a rectangular customer density function (i.e. the customers are evenly spread along the line).

We first apply Model 1 to a line of finite length which gives us Hotelling's model. The length of the market is taken as unity, and its boundaries are at 0 and 1. We refer to this type of market as a bounded one-dimensional (B, 1-D) market.

Figure 1 illustrates our definitions. Firms 1 and 2 are paired; 1 is a peripheral firm and 2 an interior firm. The boundary between 1 and 2 is located at a point $Y$ distance from the market boundary. The long side of 1's market is $Y$ (equals 1's whole market). Firm 3 is located at $3Y$. The boundary between 2 and 3 is at $2Y$. Firm 2's long side is thus $Y$ (equals its whole market) while the left hand side of 3's market is also $Y$ (its right hand size is not determined until firm 4 is located).

The necessary and sufficient conditions for equilibrium are: (1.i) no firm's whole market is smaller than any other firm's half market; (1.ii) the two peripheral firms are paired.

Any firm can capture a market equal in length to either half market of any other firm by pairing with it. Thus condition (1.i) is necessary. An unpaired peripheral firm can always increase its market by moving towards its neighbour. Thus condition (1.ii) is necessary. The proof of sufficiency is omitted for brevity but it can be found in [11, p. 9].

The application of these equilibrium conditions to various situations distinguished by the number of firms in the market is tricky. It is necessary to consider some of the cases individually.

1 Chamberlin [6, Appendix C] appears to have missed this critical condition.
One Firm. The location of one firm is indeterminate since it captures the whole market wherever it goes.

Two Firms. Both are peripheral firms and, therefore, by condition (1.ii) they must be paired. Condition (1.i) dictates that they be paired at the market's centre. This is Hotelling's $MD$ result.

Three Firms. It is impossible to satisfy the equilibrium conditions when there are three firms in the market. The only way to satisfy (1.ii) is for both peripheral firms to be paired with the interior firm. But this leaves the interior firm with a market area of virtually zero—a violation of (1.i).

Four Firms. Condition (1.ii) requires that the peripheral firms be paired and (1.i) is satisfied only if the pairs are located at the first and third quartiles.

Five Firms. The only possible equilibrium pattern for five firms is obtained by making firms 3 and $n-2$ in Figure 1 coincident—i.e. they are the same firm. The peripheral pairs are located at $1/6$ and $5/6$, and firm 3 is in the centre of the market.

Six Firms. With six firms the equilibrium configuration ceases to be unique. Two limiting cases are shown in Figure 2, (a) and (b). Both of these exhibit the necessary

*Figure 2*

symmetry shown in Figure 1. The first case, however, *minimizes* the distance between the third and fourth firms and thereby *maximizes* the market lengths of the four firms in the peripheral pairs. The second case *maximizes* the distance between 3 and 4 and thereby minimizes the equal market lengths for peripheral pairs. In the first case, all six firms have equal markets and in the second case the four firms in the peripheral pairs have markets of $1/8$ while the inner two firms have markets of $1/4$.

In equilibrium, firms 3 and 4 can be separated by any distance between the extremes of $\delta$ and $1/4$.

Firms 3 and 4 must both be separated from their neighbouring peripheral pair by the distance $2Y$, but they can be separated from each other by any distance up to $2Y$. This means that firms 3 and 4 can have any market between $1/6$ and $1/4$. The reader can easily demonstrate for himself that there is an infinite number of equilibria for each $n>5$.

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1 Lerner and Singer [16] have noted the continuance of oscillations when three firms move according to Hotelling's rules. The above treatment establishes the non-existence of equilibrium. Both Lerner and Singer and Chamberlin appear reluctant to accept this instability. Chamberlin [6] implies that two firms will be located at one quartile and the third firm at the other quartile. Equilibrium cannot, of course, exist unless the assumptions of the model are changed.

2 This is the critical result missed by Lerner and Singer [16, p. 181] in their otherwise complete analysis of this model. They do discover the extreme cases (and analyse these in detail for eight firms) but they wrongly assert that there is not a continuous range between these extremes over which the markets of the firms can vary in equilibrium. Lösch [18, p. 74], following Chamberlin [6], makes the stronger (and incorrect) assertion that "the intermediate firms are all equally spaced from each other."

3 The range of possible locations with respect to their own market boundaries is not independent of the size of their market. If they have a market of $1/4$ they must be located at its mid-point. If they have a market of only $1/6$, they must be paired and hence located at one of the ends of their market. If they are separated by a distance $d$, such that $1/6<d<1/4$ they can be located over a range within their markets.
To complete the analysis of six or more firms, three questions need to be answered. First, what is the range of possible market lengths for the various firms that is compatible with equilibrium? Second, what are the equilibrium configurations that minimize, and that maximize transport costs? Third, how do total transport costs compare with the transport costs in the socially optimal configuration?

Two propositions concerning relative market sizes follow immediately from the equilibrium conditions: (1) no firm can have a market more than twice as large as any other firm’s market; and (2) no firm can have a market smaller than \( Y \)—the market length of the firms in the peripheral pairs. The minimum and the maximum possible sizes for an individual firm’s market depend upon the number of firms in the market, and upon whether or not the firm is a member of a peripheral pair. The bounds are

\[
\frac{1}{2n-4} \leq L_p \leq \frac{1}{n} \quad \text{and} \quad \frac{1}{2n-6} \leq L_i \leq \frac{2}{n+1}
\]

where \( L_p \) is the length of the market of each of the firms in the peripheral pairs and \( L_i \) is the length of the market of any other firm.\(^1\) The configuration that minimizes transport cost for \( n \) firms has all firms spread out along the line serving equal markets of length \( 1/n \) divided into equal half-markets of \( 1/2n \). This socially optimal configuration is, however, not an equilibrium one because the peripheral firms must be paired in equilibrium. The equilibrium configuration with the lowest transport costs has two firms paired at each end of the market and all other firms spread evenly throughout the market. This configuration “wastes” the transport-cost-reducing potential of one of the two firms in each of the peripheral pairs.

The equilibrium configuration that maximizes transport costs has all firms paired (or all but one if \( n \) is odd). This configuration “wastes” the cost-reducing potential of every other firm and gives transport costs that are exactly double (\( n \) even) those resulting from the socially-optimal configuration.

We now test the basic conjecture that the behaviour of this model depends critically on the nature of the space by transferring Model 1 to a circle whose circumference is unity. This is a one-dimensional space, but if we continue to move along it in one direction or the other we do not encounter a boundary but instead return to our starting place. We refer to this market as being unbounded, finite, one-dimensional \((U, F, 1-D)\). Because there are no boundaries, there are only interior firms. Condition (1.i) is now the necessary and sufficient condition for equilibrium.

One Firm. As in the line, the location of one firm is indeterminate.

Two Firms. No matter where the second firm locates it gets half the circle as its market. Thus in contrast to \((B, 1-D)\) space, any configuration is an equilibrium one.

Three Firms. There are an infinite number of equilibria. (Contrast this with \((B, 1-D)\) space for \( n = 3 \).) To see this locate firms 1 and 2 arbitrarily and then draw a diameter

\[ L_p = \frac{1}{2n-4} \quad \text{and} \quad L_i = \frac{1}{2n-6} \]

Having established that one or more equilibria exist (for \( n \neq 3 \)), it is of interest to know whether or not the dynamic process implied by our model is convergent. This problem is considered at length in [11, footnote 16].

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\(^1\) These bounds are established as follows. Lower bound on \( L_p \): The four peripheral firms have \( Y \) each, the remaining \( n-4 \) firms have markets of \( 2Y \) (divided into half-markets of \( Y \)) giving \( Y = \frac{1}{2(n-4)+4} \). Upper bound on \( L_i \): Since no market can be smaller than \( Y \), its value is maximized by giving all firms a market of \( Y \). Given the configuration at the outer part of the market in Figure 1 we need to pair all firms if \( n \) is even and all but one if \( n \) is odd. With \( n \) even this gives \( Y = 1/n \) and with \( n \) odd \( Y = 1/(n+1) \) since the unpaired firm must have a market of \( 2Y \). (If it had less, at least one of its neighbours would have a market of less than \( Y \).) Lower bound on \( L_i \): The location of the peripheral pairs and their neighbours (as in Figure 1) determines ten half-markets (four of which are zero) and uses a distance of \( 6Y \). Now let one interior firm have a market of only \( Y \). To do this we must use another \( 2Y \) of the line and four more half-markets (two for the firm in question and one for each of its neighbours). This determines 14 half-markets (of \( Y \) or zero) and uses \( 8Y \) and leaves 1–8Y to be divided between the remaining \( 2n-14 \) half-markets. To minimize \( Y \) we maximize the markets of the other firms by giving each of them two equal half-markets of \( Y \). Thus \( Y = (1-8Y)/(2n-14) = 1/(2n-6) \). Upper bound on \( L_i \): If \( n \) is odd, pair all but one firm giving \( n+1 \) non-zero half-markets of which the firm in question has two, giving it a market of \( 2/(n+1) \). When \( n \) is even, pair all but two firms leaving \( n+2 \) non-zero half-markets, of which the non-paired one each, giving them markets of \( 2/(n+2) \).

\(^2\) Having established that one or more equilibria exist (for \( n \neq 3 \)), it is of interest to know whether or not the dynamic process implied by our model is convergent. This problem is considered at length in [11, footnote 16].
through 1 and 2 (see Figure 3) to intersect the circle at $C'$ and $C''$. Any location for firm 3 in the arc $C'C''$ produces an equilibrium configuration. Any other location puts one of the firms into disequilibrium.\(^1\)

The multiplicity of equilibria persists as \(n\) is increased and all we can do is to place some limits on the size of the firm's market. These bounds are \(1/2(n-1) \leq L \leq 2/(n+1)\), where \(L\) is the length of the market arc for a single firm.\(^2\)

On the circle, unlike the line, the socially-optimal configuration is compatible with equilibrium: all firms are equally spaced at a distance \(1/n\) apart. The cost-maximizing equilibrium configuration is the same as on the line: all firms are paired \((n\ even)\) so that transport costs are twice what they need to be for any given \(n\).

The conclusions reached in our analysis of Model 1 suggest rejection of two conjectures that are commonly found in the literature. First, the nature of the space is not critical to the behaviour of the model\(^3\): as \(n\) increases beyond 5 the behaviour on the line becomes increasingly similar to that on the circle. Second, MD is not a characteristic configuration of the linear model for \(n>2\).\(^4\)

\[\text{Figure 3}\]

\textit{Model 2.} This is the same as Model 1—a bounded linear market with a rectangular customer density function—except that the firms do not make a ZCV assumption. Instead, when any firm \(i\) chooses its location it conjectures that another firm \(j\) will relocate in such a way as to cause the maximum loss of market to \(i\)—firm \(j\) does this by pairing with \(i\) on the long side of \(i\)'s market. Firm \(i\) thus adopts a minimax strategy (MM) of choosing the location that minimizes the damage that \(j\) can do to it—i.e. it maximizes the short side of its market. The firm maximizes its short side by locating in the middle of its own market.

\(^1\) We are indebted to Professor D. Winch of McMaster University for pointing out an error in our original treatment of this problem and for suggesting this demonstration.

\(^2\) These bounds are established as follows. \textit{Lower bound on} \(L\): Make firm 1's market \(X\) by locating 2 and 3 on either side of 1. This determines four half-markets and uses \(2X\). Divide the remaining \(1-2X\) of the market into \((2n-4)\) equal half markets. Since the smallest possible whole market for 1 is one of the half markets of any of the other firms we have \(X = (1-2X)/(2n-4) = 1/2(n-1)\). \textit{Upper bound on} \(L\): This is the same as on the line.

\(^3\) Lösch [18, p. 75] for example, conjectures that it is.

\(^4\) Linear models of competition have frequently been applied to politics. Hotelling used his model to explain the similarity of the Democratic and Republican parties. Smithies motivates the introduction of downward-sloping demand curves into Hotelling's model to explain dissimilarities between parties due to concern over loss of voters at the peripheries. If the Smithies' model is extended beyond two parties, however, the implication is that the extreme parties will be the largest and all other parties will be of equal size, see [9]. To the extent that Hotelling's model (our model 1) is applicable, a variety of party sizes is possible and the extreme parties will be the smallest ones. The less extreme parties can co-exist with substantial differences in their sizes and the central party need not be the largest. In neither model will all the parties be clustered at the centre. Perhaps the most explicit development of a linear model of politics is due to Anthony Downs [8]. For a critical appraisal of such models see Donald Stokes [26].
and this implies that interior firms locate at the midpoint between their two neighbours and that peripheral firms locate one-third of the distance from the market boundary to their one neighbour.

This single proposition determines the unique equilibrium configuration for any \( n \). The firms will be spaced along the line so as to have equal market areas of \( 1/n \) and equal half markets of \( 1/2n \). Thus (with one exception) \(^1\) a minimax strategy leads the firms to locate in the socially-optimal configuration; this configuration occurs whether the firms are guarding against the damage that could be done by a new entrant or by a move from one of the existing firms.\(^2\) It is worth noting that \( MM \) is the entry minimising strategy: if \( i \) maximizes the market left to it after \( j \)'s entry, this minimizes the market available to the new entrant \( j \).

Two firms, \( j \) and \( k \), can always pair on either side of a third firm, \( i \), thus reducing \( i \)'s market to virtually zero. For this reason any \( MM \) model becomes completely indeterminate where \( n>2 \) and where firm \( i \) looks ahead to the maximally-damaging moves to be taken by two other firms.

Now transfer Model 2 to the \((U, F, 1-D)\) market of the circumference of a circle. Since peripheral firms are not paired on the bounded line, the removal of the boundaries has little effect on the behaviour of the model. There are a few differences, however, when \( n = 1 \) or 2. Unlike the line, when \( n = 1 \) the location of the firm is not determined on the circle. As with the line, the location of the two firms is determined, and is socially optimal, when they are guarding against entry by a third firm.\(^3\)

In summary, \( MM \) produces the socially-optimal configuration.\(^4\) There is no absence of equilibrium for \( n = 3 \), nor any special cases for \( n>2 \). Also the conjecture that the nature of the space critically affects the behaviour of a linear model with a \( MM \) strategy must be rejected.

\textbf{Model 3.} This is Model 1—\( ZCV \) in a \((B, 1-D)\) market—but with customer density functions that are not rectangular. We denote the customer-density function as \( c(X) \) where \( X \) is distance measured from the arbitrary origin. The firm is, of course, interested in the number of its customers rather than in the physical size of its market. In Models 1 and 2 number of customers is exactly proportional to market size. In Models 3 and 4 this is not so and the terms "size of whole market" and of "size of half market" refer to the number of customers in the relevant segments. Similarly the terms "long side" and "short side" refer not to physical length but the sides of the market containing more and less customers.

The necessary and sufficient conditions for equilibrium in Model 3 are:

\begin{enumerate}
\item[(3.i)] no firm's whole market is less than another's long-side market;
\item[(3.ii)] peripheral firms are paired;
\end{enumerate}

\(^1\) The case of two firms where the firms know there is no possibility of a third entrant provides the only exception to this rule. By locating at the centre of the market the firm ensures that it will have half of the market if its opponent pairs with it and more than half if its opponent does anything else. If one firm locates anywhere else it will have a market of less than \( 1/2 \) after the other firm has paired with it on the side that faces the centre of the whole market. If, however, the two firms are guarding against losses caused by a third firm newly entering the market, they will be driven to locate at the first and third quartiles. This gives them markets of \( 1/2 \) and they stand to have their market reduced to \( 1/4 \) by the new entrant. This distinction between a new entrant and a move by an existing firm is only relevant for two firms.

\(^2\) Michael Tietz [27] has considered a variant of Hotelling's model in which firms pursue a minimax strategy and firm 1 has \( n_1 \) plants and firm 2 has \( n_2 \), \( n_1 > n_2 \). If firm 1 locates first it can obtain more than \( n_1/(n_1 + n_2) \) of the market. This solution is not socially optimal and is in apparent contradiction to our results. Tietz' solution is, however, not an \( MM \) equilibrium since after firm 2 enters firm 1 is not in its \( MM \) equilibrium. If both firms are free to move and both are minimizing, the equilibrium is socially optimal with each plant serving \( 1/(n_1 + n_2) \) of the market.

\(^3\) If the two firms know they are playing a two-person game and are thus guarding against a move only by the other firm, their locations are indeterminate since wherever either locates, they share the market equally between themselves.

\(^4\) Samuelson [21] suggests that the socially-optimal configuration will occur on the circle only for \( n = 2, 4, 8, 16, \) etc. This is only true for Model 2 if the existing firms are not allowed to relocate after the entry of a new firm. Steven Grace [13] in reply to Samuelson, suggests a plausible dynamic of entry which implies that the number of firms in the market grow as \( 2, 4, 8, 16, \) etc.
(3.iii) if \( i \) is an unpaired interior firm \( c(B_L) = c(B_R) \);
(3.iv) if \( i \) is a paired firm \( c(B_{SS}) \geq c(B_{LS}) \);

where \( B_L \) and \( B_R \) denote the \( i \)th firm's left-hand and right-hand boundaries, and \( B_{SS} \) and \( B_{LS} \) denote its short-side and long-side boundaries.

**Proof.** Since we only need the necessity of these conditions for our subsequent analysis, we omit the proof of their sufficiency. The arguments for the necessity of (3.i) and (3.ii) are analogous to those given for Model 1. To establish the necessity of (3.iii), form the expression for the \( i \)th firm's market (\( M_i \)): when the \( i \)th firm is located at \( X_i \)

\[
M_i = \int_{B_L}^{X_i} c(x)dx + \int_{X_i}^{B_R} c(x)dx = C(B_R) - C(B_L)
\]

where \( C(X) = \int c(x)dx \). We know that \( B_L = (X_{i-1} + X_i)/2 \) and \( B_R = (X_{i+1} + X_i)/2 \). Differentiating \( M_i \) with respect to \( X_i \):

\[
\frac{\partial M_i}{\partial X_i} = \frac{1}{2}c(B_L) - \frac{1}{2}c(B_R).
\]

The first-order condition for a maximum is then \( c(B_L) = c(B_R) \) which establishes the necessity of (3.iii). To establish the necessity of (3.iv) let a paired firm consider moving away from its immediate neighbours. (By assumption it cannot move closer to the firm with which it is paired.) By the argument immediately above the rate of change of its market is

\[
\frac{\partial M_i}{\partial X_i} = \frac{1}{2}c(B_{LS}) - \frac{1}{2}c(B_{SS}).
\]

If the inequality in (3.iv) holds then \( \partial M_i/\partial X_i < 0 \) for movements away from the firm with which \( i \) is paired, and if the inequality does not hold \( \partial M_i/\partial X_i > 0 \) for such movements.

Now consider possible equilibrium configurations. Whatever the shape of the customer-density function, there is always an equilibrium for \( n = 2 \): the two firms are paired at the median of the density function. Also, there is never an equilibrium for \( n = 3 \) since the pairing of both peripheral firms violates condition (3.i) for the interior firm.

There are no equilibria for any \( n > 3 \) on a customer density function that is strictly monotonic increasing from each market boundary to a single mode! It may be helpful to consider the example of four firms locating on a symmetric uni-modal customer density function. The only configuration that satisfies (3.i) and (3.iii) has the firms paired at the quartiles of the density function. This is illustrated in Figure 4 where the areas \( R, S, T \) and \( U \) are equal. But this configuration leaves condition (3.iv) unsatisfied for the two
interior firms. Hence there is no equilibrium configuration. (The dynamic behaviour is discussed in [11], p. 20.)

We now consider variable density functions more generally and begin by establishing the following theorem.

**Theorem.** With a variable customer density function that is not rectangular over any finite range of $X$, a necessary condition for equilibrium is that the number of firms does not exceed twice the number of modes.

**Proof.** (1) Equilibrium condition (3.iii) implies that the market interval of any unpaired firm must contain at least one turning point as an interior point in the interval. Furthermore, the turning point must be a maximum (since if it were a minimum and condition (3.iii) were satisfied, the firm's market would be at a local minimum). (2) Equilibrium condition (3.iv) implies that any paired firm whose customer density is increasing away from the firm in the direction of its long-side market boundary must have a maximum point in the customer density function as an interior point in that firm's market. Since it must

![Diagram](attachment:figure5a.png)

always be true for one of any pair of firms, not located at a mode, that $c(X)$ is increasing for a movement away from the firm's location towards its long-side boundary, the long side of the market of one of the firms in each pair must include a mode. (3) Since every unpaired firm and one member of every paired firm must have a market that includes the mode as an interior point (or as a short-side boundary if the firms are paired at the mode) it is impossible to fulfill the necessary conditions with $n > 2M$.

$n \geq 2M$ is only a necessary condition. For density functions with $M > 1$ there may or may not be stable equilibrium configurations even if $4 \leq n \leq 2M$.\(^1\) A complete taxonomy would be tedious but two examples of the absence of equilibrium in bimodal distributions with $n = 4$ are illustrated in Figure 5. In each case, the firms are shown paired at the quartiles of the density function (the areas $R$, $S$, $T$, and $U$ are equal). In Figure 5(a) condition (3.iv) is not satisfied for firm $C$, while in Figure 5(b) condition (3.iv) is not satisfied for either firms $B$ or $C$.

A further point, relating to the dynamics of disequilibrium systems, is worth noting. If a firm enters the market, or considers moving, and its market is not to include the mode as an interior point it will always wish to pair with another firm. When the system is necessarily in disequilibrium ($n > 2M$) some firms must always be unable to include the

\(^1\) Downs [8] implies that a stable multi-party system requires a multi-modal distribution of voters. Our results indicate that such a system can exist with a rectangular distribution of voters, and that the number of parties that can exist in an equilibrium configuration is limited only when the distribution is non-rectangular.
mode in their markets and so will always seek to pair with another firm (who will then wish to move away from that firm). This phenomenon of pairing in disequilibrium situations is so pervasive, especially in 2-D markets, that it seems reasonable to refer to a principle of pairing as a basic characteristic of disequilibrium models.

When the number of firms is $2M$, conditions (3.iii) and (3.iv) require that the firms all be paired. Thus transport costs are necessarily twice their socially-optimal level. When $n<2M$, there is some indeterminancy in the location of the firms (at least for some density functions) and a full taxonomy of locations and transport costs does not seem worthwhile.

Now transfer Model 3 from the bounded line to the circumference of the circle. Conditions (3.i), (3.iii) and (3.iv)—but not (3.ii)—are the necessary and sufficient conditions for equilibrium on the circle. Since the proof that $n \leq 2M$ is a necessary condition for equilibrium does not employ condition (3.ii), this result generalizes immediately to the circle. Thus the change from bounded to unbounded finite space in no way affects the behaviour of the model with ZCV and a variable customer density function. The variable customer density function does, however, produce results that differ significantly from those obtained for the rectangular density function (which can now be regarded as a special case of the variable density function in which the number of modes is infinite).

**Model 4.** Model 4 combines the $MM$ strategy with a variable customer density function and is applied first to the bounded linear market. We must distinguish between a local equilibrium (no move in the neighbourhood of the firm's present location will increase its short side market) and a global equilibrium (no move will increase the firm's short side market.) There are two distinct possibilities for local equilibrium of an individual firm.

**Type I local equilibrium conditions.** The $i$th firm is located at $X_i$ such that

$$\int_{B_L}^{X_i} c(X) dX = \int_{X_i}^{B_R} c(X) dX.$$  

For interior firms

$$2c(X_i) \geq c(B_L), c(B_R) \text{ or } 2c(X_i) \leq c(B_L), c(B_R).$$

For a left-hand peripheral firm

$$2c(X_i) \geq c(B_R).$$
For a right-hand peripheral firm

\[(4.\text{ii}.c) \quad 2c(X_i) \geq c(B_L).\]

(1) If condition (4.i) does not hold, but condition (4.ii) does, the firm cannot be in equilibrium since it can always increase either one of its market sides (and hence its small side) by moving towards the boundary of the other of its market sides. (2) If condition (4.i) does hold but (4.ii) does not, the firm can increase both sides of its market by moving towards the boundary for which \(c(B) > 2c(X_i)\). (This result follows immediately from the fact that as a firm moves to the right, its right-hand market changes at the rate of \(\frac{1}{2}c(B_R) - c(X_i)\) and its left-hand market changes at the rate of \(c(X_i) - \frac{1}{2}c(B_L)\).)

Notice that Type I equilibrium locates the firm in the centre of its market—i.e. at the median of the customer density function. (This is consistent with our results for a rectangular density function.) If (4.ii) does not hold when the firm is at the centre of its market (because the density function is "too steep") the firm is not in MM equilibrium. This possibility gives rise to Type II equilibrium in which the firm does not locate at the centre of its market.

**Type II local equilibrium conditions.** An interior firm, \(i\), must satisfy either

\[(4.\text{iii}.a) \quad c(X_i) = \frac{1}{2}c(B_R) \text{ and } c(X_i) \leq \frac{1}{2}c(B_L)\]

or

\[(4.\text{iii}.b) \quad c(X_i) = \frac{1}{2}c(B_L) \text{ and } c(X_i) \leq \frac{1}{2}c(B_R).\]

A right-hand peripheral firm must satisfy

\[(4.\text{iv}.a) \quad c(X_i) = \frac{1}{2}c(B_R).\]

A left-hand peripheral firm must satisfy

\[(4.\text{iv}.b) \quad c(X_i) = \frac{1}{2}c(B_L).\]

Once these conditions hold, the firm can no longer increase its short side market by moving toward that boundary, because it is no longer true that \(c(B_S) > 2c(X_i)\).

Now consider the existence of global equilibria. Such equilibria can be shown to exist for some density functions. An example can be obtained from Figure 6 by cutting off at \(X_5\) and assuming that 3.66 is the modal point of a symmetrical distribution whose left-hand side is shown by the figure over the range \([0, X_5]\). Each firm is in type I local equilibrium. For example, firm 2's half markets are each 0.67 and firm 3's are 0.78.

Global equilibrium configurations do not always exist for any \(n\) on any density function. Figure 6 provides an example. Seven firms are located in the left-hand side of a
particular symmetrical density function in the unique configuration that satisfies the local equilibrium conditions. (The seventh firm is at the mode and the remaining six firms are not shown.) Although the local equilibrium conditions are everywhere satisfied, firm 1 is not in global equilibrium at $X_1$ since its two equal half-markets (of 0.50) are less than the two equal half markets it could obtain by locating between firms 6 and 7.\footnote{These examples lead to the conjecture that for any density function there is a maximum number of firms that is consistent with equilibrium. The conjecture is strengthened, at least for uni-modal functions, by the observation that as one moves towards higher values of $c(X)$ the number of customers served by successive firms increases—since the boundary between firms $i$ and $i+1$ must bisect the distance between them, firm $i$ must have fewer customers in each of its half markets than firm $i+1$.}

Finally, we turn to the relationship between the $MM$ equilibria and the transport-cost minimizing configuration.

**Theorem.** The $MM$ equilibrium minimizes transport costs if and only if each firm is in Type I local equilibrium.\footnote{It is well known (see e.g. [1]) that, assuming transport costs are a linear function of distance, the transport cost-minimizing location for a single firm is at the median of the density function. Our analysis generalizes this result to $n$ firms.}

**Proof.** Letting transport costs be one per unit of commodity per unit of distance, total transport costs for $n$ firms are:

$$TC = \sum_{i=1}^{n} \left\{ \int_{X_i}^{X_i+X_{i-1}} c(X)(X_i-X)dX + \int_{X_i}^{B_L} c(X)(X_i-X)dX \right\}. \quad (2)$$

There are $n$ necessary conditions for minimizing transport costs involving $n$ unknowns, $X_i$, ($i = 1, \ldots, n$) that can be generated from (2):

$$\frac{\partial TC}{\partial X_i} = 0, \quad (i = 1, \ldots, n).$$

For $i$, an interior firm, collect the terms involving $i$ in equation (2). Letting $G(X) = \int Xc(X)dX$ and $C(X) = \int c(X)dX$ and evaluating the integrals the terms can be reduced to

$$-X_{i-1}C\left(\frac{X_i+X_{i-1}}{2}\right) + 2X_iC(X_i) - 2G(X_i)$$
$$-X_{i}C\left(\frac{X_i+X_{i-1}}{2}\right) - X_{i}C\left(\frac{X_i+X_{i+1}}{2}\right) + 2G\left(\frac{X_i+X_{i+1}}{2}\right) + 2G\left(\frac{X_{i-1}+X_i}{2}\right)$$
$$-X_{i-1}C\left(\frac{X_i+X_{i+1}}{2}\right).$$

Differentiating with respect to $X_i$ and simplifying, we obtain the $i$th transport minimizing condition:

$$C(X_i) - C\left(\frac{X_i+X_{i-1}}{2}\right) = C\left(\frac{X_i+X_{i-1}}{2}\right) - C(X_i). \quad (3)$$

(3) is immediately seen to be equivalent to condition (4.i).\footnote{The demonstration that the first and $n$th transport-cost minimizing conditions are equivalent to the Type I minimax equilibrium conditions is similar to the demonstration for the $i$th interior firm, and is omitted.} Thus if all firms are in Type I equilibrium, transport-costs are minimized. If Type II equilibrium prevails for any firm, however, that firms’ half-markets are not equal. Hence the configuration is not transport-cost minimizing.

We have thus shown that for a given $n$ and a given density function, the minimax strategy may or may not lead to an equilibrium configuration, and, if it does, the configuration may or may not be transport-cost minimizing.
Model 4 can be transferred to a circle without changing any of the results that we have reached for the line: global equilibria are possible (assume for example, that Figure 6 depicts one-half of a symmetrical density function on a circle with a minimum at $X = 0$ and a maximum at $X = 3.66$ and that there are nine firms in the market); global equilibrium may not exist (assume, for example, that Figure 6 depicts one-half of a circle that goes as far as 4.66 on either side of the minimum at $X = 0$ before reaching its single maximum point, and that there are 13 firms in the market).

Conclusions for one-dimensional space

(1) The wide range of generalizations of the Hotelling model suggested by Boulding and others appears suspect. The results are very sensitive to changes in the number of firms, to changes in conjectural variation, and to changes in the distribution of customers throughout the market. Surprisingly, however, only a few of the results appear sensitive to the existence or non-existence of market boundaries.

(2) Genuine MD appears to be a very special case in the linear model, existing only for $n = 2$. This suggests that a critical step is to test the conjecture that MD will reassert itself when the market is extended to a two-dimensional space.  

(3) With a rectangular density function, ZCV produces multiple equilibria. The equilibrium set includes the socially-optimal configuration on the circle but does not on the line. With a rectangular density function, MM produces a unique equilibrium which is the socially-optimal configuration.

(4) The most surprising set of conclusions relates to the effects of abandoning the rectangular customer density function. We originally conjectured, falsely as it turned out, that the assumption of a rectangular density function was not critical on the arguments that local clusters could always be created by making the non-rectangular density function multi-modal and that, while a non-rectangular uni-modal function might pull the firms in towards the centre, it would not seriously upset any configuration established for a rectangular function. The general acceptance of some such conjecture seems necessary to explain the considerable attention that continues to be paid to rectangular customer density functions. In the ZCV models, however, equilibrium cannot exist if the number of firms exceeds twice the number of modes in the density function. Under MM, equilibrium does not necessarily exist, nor where it exists, is it necessarily socially-optimal.

III. TWO-DIMENSIONAL MARKETS

Our objectives are, of necessity, much less ambitious in 2-D space than in 1-D space both because the literature in 1-D space is more extensive and because the location problem is much simpler in 1-D space. Our two-dimensional work is limited to the effects of transferring Model 1—ZCV and a constant customer density function $[c(X, Y) = K]$—to a bounded 2-D space. We investigate the questions of existence and uniqueness of equilibrium in a space bounded by a circle, a disc.

An implication of our assumption that consumers buy from the nearest firm is that the boundary between two firms is the locus of points that are equi-distant from the two firms, and this is given by the perpendicular bisector of the line joining them.

One Firm. As with the 1-D markets, a single firm captures the whole market wherever it locates in the disc and is thus in equilibrium anywhere.

Two Firms. There is a unique equilibrium with two firms: they are paired in the centre of the market. To see this assume that firm 1 is located anywhere other than at the market centre and draw a diameter through 1. If firm 2 now enters the market and

1 A third type of 1-D space is the unbounded infinitely extensible space of the real line. A great deal of attention has been paid to such space in 2-D but not in 1-D models. We argue on p. 29 of [11] that Models 1 through 4 extend to this space with little change in their properties.

2 For an example of this conjecture see Lösch [18, p. 75] and Lipsey [17, p. 255-256].
pairs with 1 locating on the diameter and just closer to the centre than 1, firm 2 then captures more than half of the entire market. It now pays 1 to relocate on the same diameter but just inside 2, thus capturing more than half the market. If both firms are free to move they continue to "leapfrog" inwards along the diameter until they are located at the centre. At this point they split the market equally between themselves and no relocation can increase either firm's market area.

Thus two firms in the disc exactly reproduce the Hotelling result: 2's entry creates MD even if no relocation is possible, and the equilibrium with relocation produces MD with both firms located at the centre of the market.¹

Three or more Firms. If a third firm, 3, enters, when 1 and 2 are in equilibrium at the centre of the market, 3 will pair with either of the two existing firms. The firm that is paired with both of the other firms now has virtually no market and it will pay it to relocate outside of one of the other two firms. This could produce a leapfrogging outwards along a diameter that is exactly analogous to what happens with three firms in the (B, 1-D) market. In the 2-D market, however, the firms are not constrained to remain on a single diameter. Thus it is not obvious how three or more firms will behave in the disc, and using conventional analytical techniques the problem is very complex, perhaps intractable. Further analysis requires that we use a simulation technique; we conjecture an equilibrium configuration, determine its exact location, and then test the conjecture numerically.

Three configurations seemed worth investigation as candidates for equilibrium. Configuration I: All firms are evenly spaced around a circle whose radius is less than unity. Configuration II: this is the same as Configuration I except that there is an additional firm located at the centre of the disc. Configuration III: those configurations that give equilibrium in an infinitely extensible plane, i.e. a Löschian space. Elsewhere we have shown that the socially-optimal hexagonal configuration of firms is not a unique equilibrium configuration in this space (see [12]). Indeed many other configurations which give firms identical (but not necessarily regular) hexagonal, rectangular or square market areas can be equilibrium configurations in Löschian space.

Configuration I. We conjecture that the firms will be regularly spaced around a circle, concentric with the market boundary. The firms thus lie at the tips of a regular, n-sided polygon and their market areas are pieces of pie all meeting at the centre of the market. To set the firms in this configuration and check the conjecture, we need to discover the radius, r, of their circle of location. This is done as follows.

Equilibrium in any configuration requires that if any firm is free to move it will choose not to move. If Configuration I is to be an equilibrium, then if \( n - 1 \) of the firms are located at the tips of an \( n \)-sided regular polygon, defined by the circle of radius \( r \), the \( n \)th will choose to locate at the vacant tip of the polygon. \( r \) is obtained by relying on this property of equilibrium. Let \( n \) firms be so located. Rotate the axis so that the \( n \)th firm is located on the \( Y \) axis, and let this firm consider relocation, but constrain it to locate somewhere on the \( Y \) axis. Along the \( Y \) axis the \( n \)th firm's market area \( M_A \), is a function of only

¹ Nicos Devletoglou [7] develops a duopoly model in 2-D space which does not exhibit MD; in fact, it exhibits more than socially-optimal differentiation. His duopolists serve a bounded, two-dimensional market area. Each consumer is assumed to minimize travelling costs subject to the condition that he is indifferent between the two firms if the absolute value of the difference between the distances he has to travel to them is less than some critical value. This "minimum sensible" condition, or zone of indifference when combined with a "fashion" or "imitation" effect on the part of "indifferent" consumers, gives rise to uncertainty as to quantity demanded from a particular firm in a particular period, and the uncertainty implies inventory costs. This effect prevents MD because the number of such indifferent consumers (and hence inventory costs) increases as the duopolists move toward each other. It is Devletoglou's dynamic assumption, however, that is critical to the location of the duopolists. He constrains the duopolists to locate symmetrically on any diameter of the circle and effectively assumes that a move by one firm toward (away from) the centre of the market is matched by an exactly symmetrical move by the opposing duopolist, and these symmetrical reactions are anticipated. Provided demand is not completely inelastic, such a dynamic will produce the joint-profit maximizing configuration in the absence of "minimum sensible" (see [24], p. 123). In addition, the inventory costs implied by the "minimum sensible" and the fashion effect produce a dispersion of the duopolists even greater than the quartiles of the diameter.
three variables: the radius of location of the other \( n-1 \) firms; \( n \), the number of firms; and \( Z \), the distance along the \( Y \) axis at which the \( n \)th firm chooses to locate.

If our original conjecture is correct, the \( n \)th firm will want to be the same distance, \( r \), away from the origin as the other firms, provided that \( r \) is at its equilibrium value. If so we will have \( r = Z \) in equilibrium. Substituting this equality into the first order condition for a maximum of \( MA \) with respect to \( Z \) gives

\[
r = \frac{1}{2} \sqrt{\left( 1 + \sin \left( \frac{\pi}{2} - \frac{2\pi}{n} \right) \right)^2}.
\]  

(4)

If \( r \) is set at any value other than that given by the above expression, then \( \partial MA/\partial Z \neq 0 \) evaluated at \( Z = r \), and any firm would wish to move. (The details of this derivation appear in Appendix B of [11].)

To determine if the configuration is an equilibrium one with respect to a small movement of the firm in \( n \) direction is an almost impossible task using analytical methods, and in any case much more is required to establish global equilibrium. We therefore use a simulation approach. We locate \( n \) firms on a circle of radius \( r \). We then allow one firm to consider a large number of alternative locations and numerically calculate its market area for each of these.

The problem that must be solved in order to use a simulation approach to location on the disc can be simply stated: given the location of \( n \) existing firms at points \( P_i(X_i, Y_i), i = 1, \ldots, n \), what size market area can a new entrant expect to have if he locates at an arbitrarily chosen point \( P_0(X_0, Y_0) \)? Our approach is to trace out the various segments of the firm's market boundary, calculating increments to the firm's market area as we proceed around the boundary. The steps required to implement this approach are detailed in Appendix A of [11].

The market area maps produced by this technique reveal that although any firm is in a local equilibrium in Configuration I, it is not in global equilibrium. The four firm case is illustrated in Figure 7. Three firms are located on a circle of radius 0.354. The numbers in the diagram give firm 4's market area for each indicated location. (Since the disc is of unit radius, its area is \( \pi \).) The diagram shows that the point \((0, 0.354)\) is a local maximum but that it is not a global maximum. Global maxima occur at two points very close to the firm's neighbours. From \( n = 3 \) to at least 17 the same result occurs: if the firms are located on a circle of radius \( r \) and any one is free to move, it will wish to relocate next to either one of its neighbouring firms.

Thus if we impose the circular configuration, it immediately breaks up. The way in which it breaks up suggests a principle of pairing similar to that found in \((1-D)\) markets. The sub-optimal differentiation is a disequilibrium phenomenon since the other paired firm will immediately wish to shift its location.

Configuration II. One firm is located in the centre of the circle, the remaining firms are regularly spaced out around a circle of radius \( r' \). A procedure analogous to that outlined above was used to determine \( r' \). We checked this configuration for up to 17 firms and the results are as follows.

1 To determine if the point \((0, r)\) is a local equilibrium for the \( n \)th firm form the analytical expression for its \( MA \) when both the \( X \) and the \( Y \) co-ordinates can vary. Differentiate it with respect to \( X \) and \( Y \) and see if these expressions are zero evaluated at \((0, r)\). Check the second order condition to see if this is a maximum for movements that leave the firm's neighbours unchanged, discover all \((X, Y)\) combinations that satisfy the maximum conditions. For different locations, however, the firm will have different neighbours. The \( MA \) must have at least two boundaries and it can have as many as \( n+1 \) (one each with the \( n-1 \) other firms and two with the boundary of the disc). This implies there are

\[
(a+1)C_2 + (a+1)C_3 + \ldots + (a+1)C_{(a+1)} = M
\]

possible sets of potential boundaries, \( B_i \). For each \( B_i \) identify the set of points \( S_i(X_i, Y_i)(i = 1, \ldots, M) \) for which \( B_i \) is the relevant boundary. Eliminate all \( B_i \) for which \( S_i \) is the null set and for each of the remaining \( B_i \) form the expression for market area. Find the particular point within each \( S_i \) which maximizes the market area. From the set of local maximum market areas pick the largest. This yields the global maximum for one firm for given locations of the other \( n-1 \) firms.
The firm in the central location is not even at a local maximum for \( n = 3 \). For \( 3 < n < 9 \) the central firm is at a local maximum but not at a global maximum: its market is maximized by moving just outside of one of the firms on the circle. For \( n > 8 \) the central firm is in a global maximum.

The \( n - 1 \) firms located symmetrically on the circle are always at a local maximum but never at a global maximum for any \( n \) up to 17. (It did not seem worth while checking for larger values of \( n \).) Any of the \( n - 1 \) firms maximizes its market by relocating very close to either of its neighbours.

\[ \text{Configuration III.} \] The regular hexagonal configuration (which provides the most familiar equilibrium configuration in Löschan space) is adapted to the disc in the following way. Populate an infinitely extensible 2-D space with firms in a hexagonal configuration. Drop a circle centred on one firm. The firms left outside of this circle cease to exist. The first five configurations that are obtained in this manner are of 1, 7, 11, 15 and 19 firms. One firm is in equilibrium anywhere in the disc but none of the configurations for \( n > 1 \) are equilibrium configurations. For \( n = 7 \) configurations II and III are identical. Figure 8 illustrates the absence of equilibrium for \( n = 19 \). It shows the initial pattern and the configuration after one round of relocations. Clearly, the hexagonal pattern has broken up completely: there are four closely grouped pairs of firms, two groups of three, and five firms are without close neighbours. The importance of the result of this one round of relocations is that it forces us to reject the conjecture that the Löschan pattern would be only slightly distorted near the boundaries as peripheral firms squeezed in to pick up some market from their interior neighbours. This conjecture leads to only a mild distortion, but not a break-up, of the hexagonal pattern of market boundaries.
We have also transferred other patterns, such as squares and rectangles, that give equilibrium configurations in Löschian space on to the disc. The patterns always break up and the reason is always the same: firms on the periphery will prefer to pair with a neighbour rather than stay where they are.

The number of real cases for which the infinitely extensible plane is the correct analogue must be rather small and the great interest in the hexagonal configuration can only be explained by the assumption, sometimes made explicitly but more often implicitly that the results obtained from Löschian space transfer to a bounded, 2-D space.¹ Edward Leamer [15] has used simulation techniques to investigate in some detail the question of socially-optimal location patterns in (B, 2-D) space. He discovers that the socially-optimal pattern corresponds closely to the hexagonal configuration, and he presents a behavioural model of the firm which he asserts does converge to the optimal configuration. He is, however, not satisfied with his behavioural model and suggests that the model under investigation here, ZCV with respect to location, would be more appropriate. He wrongly asserts, however, that the ZCV model would also converge to the socially-optimal configuration [15, p. 242]. Beckman clearly implies that the hexagonal pattern is the equilibrium pattern for (B, 2-D) space when n is large [4, p. 41]. Samuelson seems tempted to apply the results from unbounded markets to bounded markets although he seems undecided between the square and hexagonal patterns [22, pp. 343-344].

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is mistaken. The existence of boundaries to the market is critical to the behaviour of the model in 2-D space (although not in 1-D space).

All three conjectured equilibrium configurations have been rejected and we advance the hypothesis that there is no equilibrium configuration for Model 1 on the disc for $n \geq 2$.\footnote{In a paper published in this issue Mr Shaked of Nuffield College, Oxford proves the non-existence of equilibrium for three firms on the disc.}

It is now necessary to study the dynamic behaviour of the model. We do this for two reasons: (1) there may be equilibrium configurations, the nature of which we have not guessed, but to which our dynamic model might quickly converge; (2) if we discover a pattern of perpetually recurring oscillations, we will have disproved the existence of an equilibrium configuration that is obtainable independently of initial conditions. Indeed, if our starting configuration is not just chosen haphazardly, but is in some sense a likely configuration, we will have thrown strong doubt on the possibility of ever attaining an equilibrium.

The procedure for studying the dynamic behaviour is as follows. The first firm is placed in the centre of the market and the second firm is allowed to pair with it. Each additional firm is then allowed to enter the market one at a time in its market-maximizing location. After all $n$ firms have entered, the existing firms are allowed to relocate in the sequence in which they entered.

Briefly our results are as follows. For $n = 3$ all three firms begin on a diameter through the origin which we take as the $X$ axis. They then leap-frog outwards exactly as in 1-D space. The outward movement continues to a point where it finally pays one firm to depart slightly from the $X$ axis. It then pays the next firm also to depart from the $X$ axis. The third firm then finds it most profitable to return to the centre of the disc. The other two firms immediately follow recreating the $MD$ grouping at the centre. The outward leapfrogging then begins again and the pattern repeats endlessly.

The four-firm case is shown in Figure 9. The firms enter so as to create an $MD$ configuration which may be referred to as a “main-street”. They then leapfrog out along the diameter, but the pattern soon breaks up into apparent confusion (but the principle of pairing remains clearly observable). Soon, however, it pays someone to move near the centre of the circle and the others immediately follow. They line up on a new “main-street” and the outward leapfrogging begins again. The figure shows one such sequence. After four such sequences, however, they line up in a “main-street” that exactly reproduces the initial main street.

Five firms are even more complex and we have taken the model through 70 individual moves. The firms leapfrog outwards, break up into apparent confusion, and finally regroup in a main-street near the origin. This sequence continues with each main-street configuration being near the origin but in a slightly different location than the previous one. We have not carried the dynamic model beyond $n = 5$.\footnote{This is suggestive of some urban phenomena and to study the behaviour of Model 1 with larger $n$ we allowed 23 firms to enter the market one at a time in their ZCV locations (setting $\delta$ at 0.05). The resulting configuration had 19 firms grouped in a main-street along the $X$ axis extending from $-0.45$ to $+0.45$ and two groups of two firms on the $Y$ axis, one at $0.40$ and $0.45$, and the other at $-0.40$ and $-0.45$. When relocation was allowed, the firms near the origin moved to pair with the peripheral firms on both extremes of the $X$ and $Y$ axes. The resulting configuration left no firms in the centre of the market and four local clusterings of firms well out on the positive and negative portions of the two axes. Further study of this suggestive dynamic behaviour requires the development of more sophisticated models.}

We strongly suspect, but as yet cannot prove, the non-existence of any equilibrium configurations in the disc beyond $n = 2$. Certainly, for up to $n = 17$ none of the three configurations that seemed likely to produce equilibrium actually did so. Also up to $n = 5$ there appear to be regular, cyclic oscillations.

During the whole disequilibrium process the firms tend to be clustered into several unstable groupings and all of the firms are well within the circle of location which would
*FIGURE 9*

**The Four-firm Dynamics**

The market area is a disc but only the relevant central portion is shown. The X and Y axes and the scale are shown in cell (0). In the subsequent cells only the origin is indicated by $\circ$.

1. The initial position after entry.
2. The position reached after a series of outward jumps along the X axis.
3)-(15) Successive positions reached by a single relocation of one firm. The movement of the firm in question is shown by an arrow.
minimize the costs of transport. This reinforces the conjecture that the principle of pairing (or possibly a more general principle of "local clustering") should replace the principle of minimum differentiation. It also suggests the further conjectures that the absence of equilibrium may be important in many locational contexts, and that suboptimal locations may be a persistent result through all of the dynamic fluctuations in locational patterns.

IV. CONCLUSIONS

Minimum Differentiation. Of the models that we have studied, minimum differentiation is a property only of those in which firms pursue a strategy of zero conjectural variation (ZCV) and where the number of firms is restricted to two.

Principle of Local Clustering. When a new firm enters a market, or when an existing firm relocates, there is a strong tendency for that firm to locate as close as possible to another firm. This behaviour tends to create local clusters of firms in many equilibrium and disequilibrium situations. The principle of minimum differentiation is a special case of the principle of local clustering when the number of firms in the market is restricted to two.

Multiple Equilibria. Under ZCV with a rectangular distribution of customers, an infinite number of equilibria exists in unbounded, one-dimensional space for any $n$, and in bounded one-dimensional space when $n>5$. (We showed in [12] that multiple equilibria also exist in infinitely extensible 2-D space.)

The Non-existence of Equilibrium. For $n>2$ in a bounded two-dimensional market under ZCV and an even distribution of customers there do not appear to be any equilibrium configurations. If firms adopt a ZCV strategy in a one-dimensional market, the maximum number of firms consistent with equilibrium is twice the number of modes in the density function. Even under the minimax strategy, equilibrium does not exist for some $n$ and some non-rectangular density functions.

In conclusion brief attention may be given to the sensitivity of our results to variations in those of our assumptions that may seem most restrictive.

Assumption (iv): It has seemed intuitively appealing to many readers of earlier drafts of this paper that Hotelling's minimum-differentiation result would reassert itself for $n>2$ if we assumed that customers were indifferent between two firms, whenever their delivered prices differed by less than some arbitrary amount, $\Delta$. We may briefly study this conjecture in the bounded, linear market.

Assume that firms obtain equal shares of customers who are indifferent between them and let $a$ be the cost of transporting a unit of the commodity over the whole (unit) range of the market. If all firms locate within $\Delta/2a$ of the centre of the market we will have a sort of minimum differentiation and the firms will each have $1/n$ of the total market. But unless $\Delta/a$ is large in relation to the total market this central grouping will not be an equilibrium configuration.

1 The radius of location that minimizes transport costs is 0.49, 0.56 and 0.63 for $n=3, 4$ and 5. The maximum distance from the centre of the disc reached by any of the firms during the dynamic adjustment process is 0.42, 0.45 and 0.47 for $n=3, 4$ and 5.

2 Baumol reaches a similar conclusion in a different context [2]. He considers the problem of entry in a 2-D, Lancaster type, characteristics space. When two firms are present, a third will choose to produce a product which is very similar to that produced by one of the already existing firms. For a discussion of the limitations of Baumol's approach see [23].

3 The conditions under which our results generalize to non-spatial forms of differentiation (product characteristics) are of some interest. Let the bounded line represent a continuum of some non-spatial characteristic, colour for example—and let the Cartesian co-ordinates of any point on the disc represents a combination of two characteristics, smoothness and alcoholic content of whisky, for example. Then let the customer density function describe the distribution of customer's most preferred points through the appropriate space. A firm's location is also described by the characteristic(s) of the product it produces. For our results to generalize we require that consumers buy from the firm that is nearest to their most-preferred point in the characteristic(s) space. In one-dimension this requirement is easily understood, and in two the requirement implies that a monotonie transformation of the scales on either or both of the axes can be found such that an individual's indifference curves are circular around his most preferred point.
The case of \( n = 3 \): Locate firm 1 at \( 1/2 - \Delta/2a \), firm 2 at 1/2 and firm 3 at \( 1/2 + \Delta/2a \). Let firm 1 consider moving. If it goes to \( 1/2 - \Delta/a \) it gains exclusive control of a peripheral market of \( 1/2 - \Delta/a \) and a half share of an interior market of \( \Delta/a \). It will make this move if \( 1/2 - \Delta/a + \Delta/2a > 1/3 \), i.e., if \( \Delta/a < 1/3 \). (If this condition holds it is easy to show that there is no equilibrium for \( n = 3 \).)

The case of \( n > 3 \): Locate the peripheral firms at \( 1/2 - \Delta/2a \) and \( 1/2 + \Delta/2a \) and let one interior firm consider relocating to become a peripheral firm. If it locates \( \Delta/a \) away from its nearest neighbour it obtains a peripheral market of \( 1/2 - \Delta/2a - \Delta/a \) and some part of a shared interior market of \( \Delta/a \). The firm will certainly move if \( 1/2 - \Delta/2a - \Delta/a > 1/n \), i.e. \( \Delta/a < (n-2)/3n \).

Thus if \( \Delta/a \) is large enough a sort of minimum differentiation may result although there will be a wide range (\( 1/2 - \Delta/2a \) to \( 1/2 + \Delta/2a \)) over which the firms may be haphazardly located; but if \( \Delta/a \) is not too large our results hold.

Assumption (v): It has been alleged that our model is uninteresting because we abstract from price competition. The allegation implies that if we incorporated price competition into the model, virtually all of our results would not hold. Of course there are situations in which price competition has no meaning (e.g. political behaviour). It is clear that, where price competition is relevant, at least some of our results do continue to hold. For example, the important result that peripheral firms tend to pair is not dependent upon the assumption of a parametric price (see [10] and [24]). This shows that the introduction of price competition into our models is not sufficient to establish that the socially-optimal distribution of firms is an equilibrium configuration. Precisely which of the results in this paper require alteration under price competition is difficult to establish and is the subject of further research.

Assumption (ii): Our demand assumption may seem unduly restrictive. However, many of our results are clearly not dependent upon this assumption. We have shown elsewhere [12] that the existence of multiple equilibria in unbounded 2-D space is not dependent upon the inelasticity of demand. It is well known that the tendency toward M.D. under duopoly remains when demand is less than perfectly elastic [24]. It is also true that the tendency toward pairing of peripheral firms survives the introduction of downward sloping demand (see [9] and [10]).

Assumption (vi): We have not explicitly considered entry equilibrium. Our cost assumption implies that free entry would result in one firm per customer and zero transport costs. To consider free entry we must alter the assumption to avoid this result. One common assumption is to assume that each firm has the same fixed cost of production in addition to its constant marginal cost (see [3], [9], [19] and [28]). By selecting a suitable value for the fixed costs any equilibrium configuration determined in this paper, for an arbitrary \( n \), can be made a free-entry equilibrium in the sense that existing firms are at least covering costs while a new entrant does not expect to do so. Thus any equilibrium in this paper remains a possible equilibrium given free entry.

The wide variety of theoretical results that we have obtained suggests that careful, detailed specification of the behaviour of firms, of the nature of the space, and of the distribution of customers is essential. Contrary to conjectures commonly found in the literature many of the results obtained from one model do not generalize to other models.

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