BAYES AND MINIMAX SOLUTIONS OF SEQUENTIAL DECISION PROBLEMS

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The present paper deals with the general problem of sequential choice among several actions, where at each stage the options available are to stop and take a definite action or to continue sampling for more information. There are costs attached to taking inappropriate action and to sampling. A characterization of the optimum solution is obtained first under very general assumptions as to the distribution of the successive observations and the costs of sampling; then more detailed results are given for the case where the alternative actions are finite in number, the observations are drawn under conditions of random sampling, and the cost depends only on the number of observations. Explicit solutions are given for the case of two actions, random sampling, and linear cost functions.

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SUMMARY

The problem of statistical decisions has been formulated by Wald [1] as follows: the statistician is required to choose some action $a$ from a class $A$ of possible actions. He incurs a loss $L(u, a)$, a known bounded function of his action $a$ and an unknown state $u$ of Nature. What is the best action for the statistician to take?

If $u$ is a chance variable, not necessarily numerical, with a known $a$

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priori distribution, then $E L(u, a) = R(a)$ is the expected loss from action $a$, and any action, or randomized mixture of actions, which minimizes $R(a)$ has been called by Wald a Bayes solution of the decision problem, corresponding to the given a priori distribution of $u$.

Now suppose there is a sequence $x$ of chance variables $x_1, x_2, \cdots$, whose joint distribution is determined by $u$. Instead of choosing an action immediately, the statistician may decide to select a sample of $x$'s, as this will yield partial information about $u$, enabling him to make a wiser selection of $a$. There will be a cost $c_X(x)$ of obtaining the sample $x_1, \cdots, x_N$ and, in choosing a sampling procedure, the statistician must balance the expected cost against the expected amount of information to be obtained.

Formally, the possibility of making observations leaves the situation unchanged, except that the class $A$ of possible actions for the statistician has been extended. His action now consists of choosing a sampling procedure $T$ and a decision function $D$ specifying what action $a$ will be taken for each possible result of the experiment. The expected loss is now $R(T, D) = l(T, D) + c(T)$, where $l(T, D)$ is the expected value of $L(u, a)$ for the specified sampling procedure and decision rule, and $c(T)$ is the expected cost of the sampling procedure. A Bayes solution is now a pair $(T, D)$, or randomized mixture of pairs $(T, D)$ for which $R(T, D)$ assumes its minimum value.

The minimizing $T = T^*$ has been implicitly characterized by Wald, and may be described by the rule: at each stage, take another observation if and only if there is some sequential continuation which reduces the expected risk below its present level. The main difficulty here is that various quantities which arise are not obviously measurable: for instance if the first observation is $x_1$, we must compare our present risk level, say $w_1(x_1)$, with $z(x_1) = \inf w(x_1, T, D)$, where $w(x_1, T, D)$ is the expected


Many of the results in this paper overlap with those obtained previously by A. Wald and J. Wolfowitz [4], and also with some prior unpublished results of Wald and Wolfowitz, announced by Wald at the meeting of the Institute of Mathematical Statistics at Berkeley, California, June 22, 1948. Sections 3 and 6 of the present paper contain analogues of Lemmas 1–4 of [4], though both the statements and the proofs differ because of the generally different approach. The proof that the sequential probability ratio test of a dichotomy minimizes the expected number of observations under either hypothesis, in Section 5 of the present paper, follows from Section 3 in the same way that the proof of the same theorem follows from Lemmas 1–8 in [4]. The previously mentioned unpublished results of Wald and Wolfowitz include the main result of Section 2 (structure of the optimum sequential procedure for the finite multi-decision problem) in the special case of linear cost functions.
risk for any possible continuation \((T, D)\); we take another observation if and only if \(w_1 > z\). It is not a priori clear that \(z\) will be a measurable function of \(x_1\), so that the set of points \(x_1\) for which we stop may not be measurable. Actually, \(z\) always is measurable, as we shall show.

A characterization of the minimizing \(T = T^*\) is obtained for hypotheses involving a finite number of alternatives under the condition of random sampling. It consists of the following: We are given \(k\) hypotheses \(H_i\) \((i = 1, 2, \ldots, k)\) which have an a priori probability \(g_i\) of occurring, a risk matrix \(W = (w_{ij})\) where \(w_{ij}\) represents the loss incurred in choosing \(H_j\) when \(H_i\) is true, and a function \(c(n)\) which represents the cost of taking \(n\) observations. It is shown that for each sample size \(N\), there exist \(k\) convex regions \(S_i^*\) in the \((k - 1)\)-dimensional simplex spanned by the unit vectors in Euclidean \(k\)-space whose boundaries depend on the hypotheses \(H_i\), the risk matrix \(W\) and the cost function \(c_N(n) = c(N + n) - c(n)\). These regions have the property that if the vector \(\bar{g}(N)\) whose components represent the a posteriori probability distribution of the \(k\) hypotheses lies in \(S_i^*\), the best procedure is to accept \(H_j\) without further experimentation. However, if \(\bar{g}(N)\) lies in the complement of \(\sum_{j=1}^{k} S_j^*\), the best procedure is to continue taking observations. At any stage, the decision whether to continue or terminate sampling is uniquely determined by this sequence of \(k\) regions and moreover this sequence of regions completely characterizes \(T^*\).

A method for determining the boundaries of these convex regions is given for \(k = 2\) (dichotomy) when the cost function is linear. It is shown that in this special case, \(T^*\) coincides with Wald's sequential probability ratio test.

The minimax solution to multi-valued decision problems is considered and methods are given for obtaining them for dichotomies. It is shown that in general, the minimax strategy for the statistician is pure, except when the hypotheses involve discrete variates. In the latter case, mixed strategies will be the rule rather than the exception.

Examples of double dichotomies, binomial dichotomies, and trichotomies are given to illustrate the construction of \(T^*\) and the notion of minimax solutions.

It may be remarked that the problem of optimum sequential choice among several actions is closely allied to the economic problem of the rational behavior of an entrepreneur under conditions of uncertainty. At each point in time, the entrepreneur has the choice between entering into some imperfectly liquid commitment and holding part or all of his funds in cash pending the acquisition of additional information, the latter being costly because of the foregone profits.

\(^2\) The possibility of nonmeasurability is not considered in [1] or [4].
1. CONSTRUCTION OF BAYES SOLUTIONS

The Decision Function

We have seen that the statistician must choose a pair \((T, D)\). It turns out that the choice of \(D\) is independent of that of \(T\):

**Lemma:** There is a fixed sequence of decision functions \(D_m\) such that

\[
R(T, D_m) \to \inf R(T, D) = w(T) \text{ for all } T.
\]

This will be the main result of this section. It follows that the expected loss from a procedure \(T\) may be taken as \(w(T)\), since this loss may be approximated to arbitrary accuracy by appropriate choice of \(D_m\), and a best sequential procedure \(T^*\) of a given class will be one for which \(w(T^*) = \inf w(T)\) where the inf is taken over all procedures \(T\) of the class under consideration.

We are considering, then, a chance variable \(u\) and a sequence \(x\) of chance variables \(x_1, x_2, \ldots\). A sequential procedure \(T\) is a sequence of disjunct sets \(S_0, S_1, \ldots, S_N, \ldots\), where \(S_N\) depends only on \(x_1, \ldots, x_N\) and is the event that the sampling procedure terminates with the sample \(x_1, \ldots, x_N\); we require that \(\sum_{N=0}^{\infty} P(S_N) = 1\). \(S_0\) is the event that we do not sample at all, but take some action immediately; it will have probability either 0 or 1.

A decision function \(D\) is a sequence of functions \(d_0, d_1(x_1), \ldots, d_N(x_1, \ldots, x_N), \ldots\), where each \(d_N\) assumes values in \(A\), and specifies the action taken when sampling terminates with \(x_1, \ldots, x_N\). We admit only decision functions \(D\) such that \(L[u, d_N(x)]\) is for each \(N\) a measurable function.

**Proof of Lemma:** The loss from \((T, D)\) is \(G(u, x; T, D) = L[u, d_N(x)] + c_N(x)\) for \(x \in S_N\), and \(\mathbb{E}G = R(T, D)\). Here, \(c_N(x)\) depends only on \(x_1, \ldots, x_N\). Then, denoting by \(\mathbb{E}_N\) the conditional expectation given \(x_1, \ldots, x_N\), we have \(\mathbb{E}_N G = \mathbb{E}_N L[u, d_N(x)] + c_N(x)\) for \(x \in S_N\), and

\[
R(T, D) = \sum_{N=0}^{\infty} \int_{S_N} \mathbb{E}_N L(u, d_N) \, dP + c(T).
\]

Now fix \(N\); we shall show that we can choose a sequence of functions \(d_{N_m}(x), m = 1, 2, \ldots\), such that

(a) \(\mathbb{E}_N L(u, d_{N_m}) \geq \mathbb{E}_N L(u, d_{N,m+1})\) for all \(x\),

(b) \(\mathbb{E}_N L(u, d_N) \geq r_N\) for all \(d_N\) and all \(x\), where

\[
r_N(x) = \lim_{m \to \infty} \mathbb{E}_N L(u, d_{N_m}).
\]

(c) \(r_N \geq \mathbb{E}_N r_n\) if \(n \geq N\).
First choose a sequence \( d'_{N,m} \) such that
\[
\delta L(u, d'_{N,m}) \to \inf_{d_N} \delta L(u, d_N) = r.
\]
Now define \( d_{N,m} \) inductively as follows: \( d_{N,1} = d'_{N,1} \); \( d_{N,m} = d'_{N,m} \) for those values of \( x \) such that \( \delta L_N(u, d'_{N,m}) \leq \delta L_N(u, d_{N,m-1}) \), otherwise \( d_{N,m} = d_{N,m-1} \). Then certainly (a) holds, so that \( \lim_{m \to \infty} \delta L_N(u, d_{N,m}) = r_N(x) \) exists. Also \( \delta L_N(u, d_{N,m}) \leq \delta L_N(u, d'_{N,m}) \), so that \( \delta r_N = r \). Choose any \( d_N \) and any \( \delta > 0 \), and let \( S \) be the event \( \{ \delta L_N(u, d_N) < r_N(x) - \delta \} \).
Then, defining \( d_{N,m}^* = d_N \) on \( S \), \( d_{N,m}^* = d_{N,m} \) elsewhere, we have
\[
\delta L(u, d_{N,m}^*) \leq \int_S r_N(x) \, dP + \int_{C_S} \delta L_N(u, d_{N,m}) \, dP - \delta P(S),
\]
so that \( \lim_{m \to \infty} \delta L(u, d_{N,m}^*) \leq r - \delta P(S) \), and \( P(S) = 0 \). This establishes (b). Finally, (c) follows from the fact that every \( d_N(x) \) is also a possible \( d_n(x) \) if \( n > N \). This means that, defining \( d^*_n = d_N \), we have \( \delta L_N(u, d_n) = \delta L_N[u, d_n(u, d^*_n)] \geq \delta L_N r_n \) for all \( d_n \), and consequently (c) holds.

Now define \( D_m = \{ d_{N,m} \} \). Since \( \delta L_N(u, d_{N,m}) \) decreases with \( m \), (1.2) yields that
\[
R(T, D_m) \to \sum_{N=0}^{\infty} \int_{S_N} r_N(x) \, dP + c(T) = w(T),
\]
and, using (b), that \( R(T, D) \geq w(T) \) for all \( D \). Thus we have reduced the problem of finding Bayes solutions to the following: we are given a sequence \( x \) of chance variables \( x_1, x_2, \ldots \), and a sequence of non-negative expected loss functions \( w_0, \ldots, w_N = r_N(x_1, \ldots, x_N) + c_N(x_1, \ldots, x_N) \). \( c_N \) is the cost of the first \( N \) observations, and \( r_N \) is the loss due to incomplete information. With each sequential procedure \( T = \{ S_N \} \) there is associated a risk \( w(T) = \sum_N \int_{S_N} w_N(x) \, dP \).

How can \( T \) be chosen so that \( w(T) \) is as small as possible?

The Best Truncated Procedure

Among all sequential procedures not requiring more than \( N \) observations, there turns out to be a best, i.e., one whose expected risk does not exceed that of any other. Moreover, the procedure can be explicitly described, by induction backwards, in such a way that its measurability is clear. After \( N - 1 \) observations \( x_1, \ldots, x_{N-1} \), we compare the present risk \( w_{N-1} \) with the conditional expected risk \( \delta L_{N-1} w_N \) if we take the final observation. Thus, by choosing the better course, we can limit our loss to \( \alpha_{N-1} = \min(w_{N-1}, \delta L_{N-1} w_N) \), which may be considered as the attainable
risk with the observations $x_1, \cdots, x_{N-1}$. We can then decide, on the basis of $N - 2$ observations, whether the $(N - 1)$st is worth taking by comparing the present risk, $w_{N-2}$, with $\xi_{N-2}x_{N-1}$, the attainable risk if $x_{N-1}$ is observed. Continuing backwards, we obtain at each stage an expected attainable risk $\alpha_k$ for the observations $x_1, \cdots, x_k$, and a description of how to attain this risk, i.e., of when to take another observation. This is formalized in the following:

**Theorem:** Let $x_1, \cdots, x_N; w_0, \cdots, w_N$ be any chance variables, $w_i = w_i(x_1, \cdots, x_i)$. Define $\alpha_N = w_N$, $\alpha_j = \min (w_j, \xi_j\alpha_{j+1})$ for $j < N$, $S_i = \{w_i > \alpha_i$ for $i < j, w_j = \alpha_j\}$. Then for any disjoint events $B_0, \cdots, B_N, B_i$ depending only on $x_1, \cdots, x_i$, $\sum_{i=0}^{N} P(B_i) = 1$, we have

$$\sum_{i=0}^{N} \int_{S_i} w_i \, dP \leq \sum_{i=0}^{N} \int_{B_i} w_i \, dP.$$  

**Proof:** We shall show that, for fixed $i$ and any $(x_1, \cdots, x_i)$-set $A$,

$$\sum_{j \geq i} \int_{A S_j} \alpha_j \, dP = \sum_{j \geq i} \int_{A S_j} \alpha_i \, dP,$$

and that, for fixed $j$, and any disjoint sets $A_j, \cdots, A_N$ with $A_i$ depending only on $x_1, \cdots, x_i$ and $\sum_{i=j+1}^{N} A_i$ depending only on $x_1, \cdots, x_j$,

$$\sum_{i \geq j} \int_{A_i} \alpha_i \, dP \leq \sum_{i \geq j} \int_{A_i} \alpha_i \, dP.$$  

Choosing $A = B_i$ in (1.3) and summing over $i$, choosing $A_i = B_i S_j$ in (1.4) and summing over $j$, and adding the results yields

$$\sum_{i,j=0}^{N} \int_{B_i S_j} \alpha_i \, dP \leq \sum_{i,j=0}^{N} \int_{B_i S_j} \alpha_i \, dP.$$  

Now on $S_j$, $\alpha_i = w_j$, and always $\alpha_i \leq w_i$. Making these replacements in (1.5) yields the theorem.

We now prove (1.3) and (1.4). The relationship (1.3) is clear for $i = N$; for $i < N$,

$$\sum_{i \geq j} \int_{A S_j} \alpha_j \, dP = \int_{A S_j} \alpha_i \, dP + \int_{A S_{i+1} + \cdots + S_N} \alpha_i \, dP.$$  

But on $S_{i+1} + \cdots + S_N$, $\alpha_i = \xi_i\alpha_{i+1}$; making this replacement in the final integral and using induction backward on $i$ completes the proof.

The relationship (1.4) is clear for $j = N$; for $j < N$,

$$\sum_{i \geq j} \int_{A_i} \alpha_j \, dP = \int_{A_{i+1} + \cdots + A_N} \alpha_{j+1} \, dP \leq \sum_{i \geq j} \int_{A_{i+1} + \cdots + A_N} \alpha_{j+1} \, dP.$$
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where the inequality is obtained from the fact that always $\alpha_i \leq \xi_{ij} \alpha_{i+1}$. An induction backward on $j$ now completes the proof of (1.4).

The Best Sequential Procedure

We are given now a sequence of functions $w_0, w_1, \ldots, w_N, \ldots$, where $w_N = r_N(x_1, \ldots, x_N) + c_N(x_1, \ldots, x_N)$. The sequence $r_N(x)$ is uniformly bounded, since we supposed the original loss function $L(u, a)$ to be bounded, and we have shown that $r_N \geq \xi_n r_n$ for $n > N$. We shall suppose that $c_N(x)$ is a nondecreasing sequence, $c_N(x) \to \infty$ as $N \to \infty$ for all $x$. We now construct a best sequential procedure.

The best sequential procedure is obtained as a limit of the best truncated procedures given in the preceding section.

We first define $\alpha_{N+1} = w_N, \alpha_{i+1} = \min (w_i, \xi_{i} \alpha_{i+1})$, $S_j = \{w_i > \alpha_{i+1} \text{ for } i < j, w_j = \alpha_{j+1}\}$. For fixed $j$, $\alpha_{i+1}$ is a decreasing sequence of functions; say $\alpha_{i+1} \to \alpha$ as $N \to \infty$. Then $\alpha_j = \min (w_j, \xi_{j} \alpha_{j+1})$. Define $S_j = \{w_i > \alpha_i \text{ for } i < j, w_j = \alpha_j\}$. We shall prove that $T^* = \{S_j\}$ is a best sequential procedure, i.e., $T^*$ is a sequential procedure, and for any sequential procedure $T = \{B_i\}$,

$$w(T^*) = \sum_{i=0}^{\infty} \int_{S_j} w_i \ dP \leq \sum_{i=0}^{\infty} \int_{B_i} w_i \ dP = w(T).$$

Now

$$\sum_{i=0}^{\infty} \int_{B_i} w_i \ dP \geq \sum_{i > N} \int_{B_i} c_N \ dP \geq \sum_{i > N} \int_{B_i} w_N \ dP - M \sum_{i > N} P(B_i),$$

where $M$ is the uniform upper bound of $r_1(x), r_2(x), \ldots$. Thus

$$\sum_{i=0}^{\infty} \int_{B_i} w_i \ dP + \int_{B_N} w_N \ dP + \sum_{i > N} \int_{B_i} w_N \ dP \leq w(T) + MP(B_{N+1} + \cdots),$$

so that $w(T_N) \to w(T)$, where $T_N$ is the truncated test $B_0, \cdots, B_{N-1}, B_N + B_{N+1} + \cdots$. From the preceding section,

$$\sum_{j=0}^{K} \int_{S_{jN}} w_j \ dP \leq w(T_N)$$

for all $K$. Then

3 The assumption made here is somewhat weaker than Condition 6 in [1], p. 297. The only other assumption made, that $L(u, a)$ is bounded, is Condition 1 in [1], p. 297.
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\[ \sum_{j=0}^{K} \int_{S_j} w_j \, dP \leq w(T), \]

letting \( N \to \infty \), and using Lebesgue's convergence theorem and the easily verified fact that the characteristic function of \( S_{jN} \) approaches that of \( S_j \), and \( w(T^*) \leq w(T) \).

It remains to prove that \( T^* \) really is a sequential test, i.e.,

\[ \sum_{N=0}^{\infty} P(S_N) = 1. \]

Write \( A_N = C(S_0 + \cdots + S_N), \prod_{N=1}^{\infty} A_N = A \); we show that \( P(A) = 0 \).

It is easily verified by induction that \( \alpha_{jN} \geq c_i \) for all \( N \). The relation (1.3), with \( i = 0 \), \( A = \) sample space, yields that

\[ \alpha_{0N} \geq \sum_{j=0}^{N} \int_{S_jN} c_m \, dP = \int_{C(S_0N + \cdots + S_mN)} c_m \, dP \]

for all \( m < N \). Then

\[ \alpha_0 \geq \int_{A_m} c_m \, dP \geq \int_A c_m \, dP \]

for all \( m \). Since \( c_m \to \infty \), \( P(A) = 0 \).

We now prove that \( w(T^*_N) = \alpha_0 \). If \( T_N^* \) denotes the truncated test \( S_0, S_1, \cdots, S_N + S_{N+1} + \cdots \), the proof that \( w(T_N^*) \to w(T) \) shows that \( w(T^*_N) \to w(T^*) \). Also (1.3), with \( i = 0 \), \( A = \) sample space, shows that

\[ \alpha_{0N} = \sum_{j=0}^{N} \int_{S_jN} w_j \, dP. \]

Since \( \{S_{0N}, \cdots, S_{NN}\} \) is the best of all procedures truncated at \( N \), and \( T^* \) is the best of all sequential procedures, \( w(T^*) \leq \alpha_{0N} \leq w(T^*_N) \). Letting \( N \to \infty \) yields \( w(T^*) = \alpha_0 \).

Now \( S_0 = \{w_0 \leq \alpha_0\} \); i.e., the best procedure \( T^* \) is to take no observations if and only if there is no sequential procedure which reduces the risk below its present level. This remark, which identifies our procedure with that characterized by Wald, at least at the initial stage, will be useful in the next section.

2. Bayes Solutions for Finite Multi-Valued Decision Problems

In this section we shall seek a characterization of the optimum sequential procedure developed in section 1 in cases where the number of alter-
native hypotheses is finite. It will be shown that the optimum sequential test for a \( k \)-valued decision problem is completely defined by \( k \) (or a sequence of \( k \)) convex regions in a \( (k - 1) \)-dimensional simplex spanned by the unit vectors. No procedure has yet been developed for determining the boundaries of these regions in the general case. However, for \( k = 2 \) (dichotomy) and for a linear cost function, a method for determining the two boundaries has been found and the optimum test is shown to be the sequential probability ratio test developed by Wald [2].

**Statement of the Problem**

We are given \( k \) hypotheses \( H_1, H_2, \ldots, H_k \) where each \( H_i \) is characterized by a probability measure \( u_i \) defined over an \( R \) dimensional sample space \( E_R \) and has an a priori probability \( g_i \) of occurring. We are also given a risk matrix \( W = (w_{ij}) \), \( (i, j = 1, 2, \ldots, k) \), where \( w_{ij} \) is a non-negative real number and represents the loss incurred in accepting the hypothesis \( H_j \) when in fact \( H_i \) is true. (We shall assume that \( w_{ii} = 0 \) for all \( i \). This is based on the supposition, which appears reasonable, that the decision maker is not to be penalized for selecting the correct alternative, no matter how unpleasant its consequences may be.) In addition to the risk matrix \( (w_{ij}) \) we shall assume that the cost of experimentation depends only on the number \( (n) \) of observations taken and is given by a function \( c(n) \) which approaches infinity as \( n \) approaches infinity. The problem is to characterize the procedure for deciding on one of the \( k \) alternative hypotheses which results in a minimum average risk. This risk is defined as the average cost of taking observations plus the average loss resulting from erroneous decisions.

**Structure of the Optimum Sequential Procedure**

Let \( G_k \) stand for the convex set in the \( k \) dimensional space defined by the vectors \( \bar{g} = (g_1, g_2, \ldots, g_k) \) with components \( g_i \geq 0 \) and \( \sum_{i=1}^k g_i = 1 \); and let \( H = (H_1, H_2, \ldots, H_k) \) represent the \( k \) hypotheses under consideration. Then every vector \( \bar{g} \) in \( G_k \) may be considered as a possible a priori probability distribution of \( H \).

For any \( \bar{g} \) in \( G_k \) and for any sequential procedure \( T \), (see definition in Section 1), let \( R(\bar{g} \mid T) \) represent the average risk entailed in using the test \( T \) when the a priori distribution of \( H \) is \( \bar{g} \). Then

\[
R(\bar{g} \mid T) = \sum_{i=1}^k g_i \mathbb{E}_i[c(n) \mid T] + \sum_{i=1}^k \sum_{j=1}^k g_i w_{ij} P_{ij}(T),
\]

where \( \mathbb{E}_i[c(n) \mid T] \) is the average cost of observations when the sequential test \( T \) is used and \( H_i \) is true, and \( P_{ij}(T) \) is the probability that the sequential test \( T \) will result in the acceptance of \( H_j \) when \( H_i \) is true. The
risk involved in accepting the hypothesis \( H_i \) prior to taking any observations will be designated by \( R_j (j = 1, 2, \ldots, k) \) and is given by

\[
R_j = \sum_{i=1}^{k} g_i w_{ij}.
\]

We now define \( k \) subsets \( S^*_1, S^*_2, \ldots, S^*_k \) of \( G_k \) as follows: A vector \( \vec{g} \) of \( G_k \) will be said to belong to \( S^*_j \) if (a) \( \min (R_1, R_2, \ldots, R_k) = R_j \) and (b) \( R(\vec{g} | T) \geq R_j \) for all \( T \). We observe that since the unit vector with 1 in the \( j \)th component belongs to \( S^*_j \), the subsets \( S^*_j \) are nonempty. We now prove the following

**Theorem.** The sets \( S^*_j \) are convex. That is, if \( \vec{g}_1 \) and \( \vec{g}_2 \) belong to \( S^*_j \) so does \( \vec{g} = a\vec{g}_1 + (1 - a)\vec{g}_2 \) for all \( a, 0 \leq a \leq 1 \).

**Proof:** Assume the contrary. Then there exists a sequential procedure \( T \) such that

\[
R(\vec{g} | T) = \sum_{i=1}^{k} g_i \xi_i [c(n) | T] + \sum_{i=1}^{k} \sum_{j=1}^{k} g_i w_{ij} P_{ij} < \sum_{i=1}^{k} g_i w_{ij}.
\]

But by definition, if either \( \vec{g}_1 \) or \( \vec{g}_2 \) represents the a priori distribution of the hypotheses \( H \), we must have for all sequential procedures and hence for \( T \),

\[
R(\vec{g}_1 | T) = \sum_{i=1}^{k} g_i \xi_i [c(n) | T] + \sum_{i=1}^{k} \sum_{j=1}^{k} g_i w_{ij} P_{ij} \geq \sum_{i=1}^{k} g_i w_{ij},
\]

and

\[
R(\vec{g}_2 | T) = \sum_{i=1}^{k} g_{2i} \xi_i [c(n) | T] + \sum_{i=1}^{k} \sum_{j=1}^{k} g_{2i} w_{ij} P_{ij} \geq \sum_{i=1}^{k} g_{2i} w_{ij}.
\]

If we now multiply (2.4) by \( a \) and (2.5) by \( (1 - a) \) and add, we see that the resulting expression contradicts (2.3). This proves the theorem.\(^4\)

It is easily seen that for given hypotheses \( H \) the shape of the convex regions \( S^*_j \) will depend on the cost function \( c(n) \) and the risk matrix \( W \). Thus if the cost of taking a single observation were prohibitive, the region \( S^*_j \) in \( G_k \) would simply consist of all vectors \( \vec{g} \) for which \( \min (R_1, R_2, \ldots, R_k) = R_j \). On the other hand, if the cost of taking observations were negligible and the risk of making an erroneous decision large, the regions \( S^*_j \) would shrink to the vertices of the polyhedron \( G_k \). To exhibit the dependence of the regions \( S^*_j \) on \( H, c(n) \), and \( W \), we shall use the symbol \( S^*_j [H, c(n), W] \). We shall also use the symbol \( S^* [H, c(n), W] \)

\(^4\) A similar proof shows the convexity of the corresponding regions in cases where the number of alternatives is infinite.
to represent the region consisting of all vectors \( \vec{g} \) in \( G_k \) which belong to the complement of \( \sum_{j=1}^{k} S^*_j[H, c(n), W] \).

We now define \( c_N(n) = c(N + n) - c(N) \) for all \( N = 0, 1, 2, \cdots \). Thus \( c_N(n) \) represents the cost of taking \( n \) observations when \( N \) observations have already been taken.

We shall now show that, for random sampling, the problem of characterizing the optimum sequential procedure \( T^* \) for a given \( H, c(n) \) and \( W \) reduces itself to the problem of finding the boundaries of the regions \( S^*_j[H, c_N(n), W] \) for all \( N \). The truth of this can be seen from the following considerations:

We are initially given a vector \( \vec{g} \) in \( G_k \) as the a priori distribution of the hypotheses \( H \). Initially we are also given a matrix \( W \) and a function \( c(n) = c_0(n) \). Now assume we have taken \( N \) independent observations \((N = 0, 1, 2, \cdots, )\). These \( N \) observations transform the initial state into one in which (a) the vector \( \vec{g} \) goes into a vector \( \vec{g}^{(N)} \) in \( G_k \) where each component \( g_i^{(N)} \) of \( \vec{g}^{(N)} \) represents the new a priori probability of the hypothesis \( H_i \) (i.e., the a posteriori probability of \( H_i \) given the values of the \( N \) observations), (b) the risk matrix \( W \) remains unchanged, and (c) the cost function \( c(n) \) goes into the function \( c_N(n) \).

Assume now that the boundaries of the regions \( S^*_j[H, c_N(n), W] \) are known for each \( j \) and \( N \). Then, if we take the observations in sequence, we can determine at each stage \( N(N = 0, 1, 2, \cdots, ) \) in which of the \( k + 1 \) regions the vector \( \vec{g}^{(N)} \) lies. If \( \vec{g}^{(N)} \) lies in \( S^*[H, c_N(n), W] \), then, by definition of this region, there exists a sequential test \( T \) which, if performed from this stage on, would result in a smaller average risk than the risk of stopping at this stage and accepting the hypothesis corresponding to the smallest of the quantities \( R_j^{(N)} = \sum_{i=1}^{k} g_i^{(N)} w_{ij} (j = 1, 2, \cdots, k) \). But it has been shown in section 1 that if any sequential test \( T \) is worth performing, the optimum test \( T^* \) is also worth performing. Now \( T \) will coincide with \( T^* \) for at least one additional observation. But when that observation is taken \( \vec{g}^{(N)} \) will become \( \vec{g}^{(N+1)} \) and \( c_N(n) \) will become \( c_{N+1}(n) \). Again if \( \vec{g}^{(N+1)} \) lies in \( S^*[H, c_{N+1}(n), W] \), the same argument will show that it is worth taking another observation. However, if \( \vec{g}^{(N+1)} \) lies in \( S^*_j[H, c_{N+1}(n), W] \) for some \( j \), it implies that there exists no sequential test \( T \) which is worth performing and hence the optimum procedure is to stop sampling and accept \( H_j \).

Thus we see that the optimum sequential test \( T^* \) is identical with the following procedure: Let \( N = 0, 1, 2, \cdots, \) represent the number of observations taken in sequence. For each value of \( N \) we compute the vector \( \vec{g}^{(N)} \) representing the a posteriori probabilities of the hypotheses \( H \). As long as \( \vec{g}^{(N)} \) lies in \( S^*[H, c_N(n), W] \) we take another observation. We stop sampling and accept \( H_j \) \( (j = 1, 2, \cdots, k) \) as soon as \( \vec{g}^{(N)} \) falls in the region \( S^*_j[H, c_N(n), W] \).
We have as yet no general method for determining the boundaries of \( S_j^*[H, c_N(n), W] \) for arbitrary \( H, c_N(n) \) and \( W \). However, in the case of a dichotomy \( (k = 2) \) and a linear cost function, such a method has been found and will be discussed here in detail. We shall also give below some illustrative examples of the optimum sequential test for trichotomies \( (k = 3) \).

3. Optimum Sequential Procedure for a Dichotomy When the Cost Function is Linear

We are given two alternative hypotheses \( H_1 \) and \( H_2 \), which, for the sake of simplicity, we assume are characterized respectively by two probability densities \( f_1(x) \) and \( f_2(x) \) of a random vector \( X \) in an \( R \)-dimensional Euclidean space. (If \( X \) is discrete \( f_1(x) \) and \( f_2(x) \) will represent the probability under the respective hypotheses that \( X = x \)). We assume that the a priori probability of \( H_1 \) is \( g \) and that of \( H_2 \) is \( 1 - g \), where \( g \) is known. (Later we shall show how to construct the minimax sequential procedure whose average risk is independent of \( g \)). We are also given two nonnegative numbers \( W_{12} \) and \( W_{21} \) where \( w_{ij}(i \neq j = 1, 2) \) represents the loss incurred in accepting \( H_j \) when \( H_i \) is true. In addition we shall assume that the cost per observation is a constant \( c \) which, by a suitable change in scale, can be taken as unity. We also assume that the observations taken during the course of the experiment are independent. We define \( P_{1n} = \prod_{i=1}^{n} f_1(x_i) \) and \( P_{2n} = \prod_{i=1}^{n} f_2(x_i) \) where \( x_1, x_2, \ldots, x_n \) represent the first, second, etc., observation.

If we apply the discussion of Section 2 to the dichotomy under consideration we see that the convex regions \( S_j^*[j = 1, 2] \) reduce themselves to two intervals, \( I_1 \) and \( I_2 \) where \( I_1 \) consists of points \( g \) such that \( 0 \leq g \leq g \) and \( I_2 \) consists of points \( g \) such that \( g \leq g \leq 1 \) where \( 0 \leq g \). Moreover, in view of the assumption of constant cost of observations, the boundaries \( g \) and \( g \) of these two intervals are independent of the number of observations taken but depend only on \( w_{12} \) and \( w_{21} \) (and, of course \( c \), which is taken as unity).\(^6\)

The intervals \( I_1 \) and \( I_2 \) have the following properties: If the a priori probability \( g \) for \( H_1 \) belongs to \( I_1 \), then there exists no sequential procedure which will result in a smaller average risk than the risk \( R_1 = w_{12}g \) of accepting \( H_2 \) without further experimentation. If the a priori probability \( g \) for \( H_1 \) belongs to \( I_2 \), there exists no sequential procedure which will result in a smaller average risk than the risk \( R_2 = w_{21}(1 - g) \) of accepting \( H_1 \) without any further experimentation. However, in case

\(^6\) It is assumed here that the intervals are closed; this assumption has not yet been justified. It will be shown below (Section 3) that it is a matter of indifference whether the endpoints are included or not.
$g < g < \tilde{g}$, then there exists a sequential test $T$ whose average risk will be less than the minimum of $R_1$ and $R_2$.

Using the argument of Section 2, we see that if in the initial stage $g \leq g$, the optimum procedure is to accept $H_2$ without taking any observations. Similarly, if in the initial stage $g \geq \tilde{g}$, the optimum procedure is to accept $H_1$ without taking any observations. However, if $g < g < \tilde{g}$ then there exists a sequential test $T$ worth performing and this test will coincide with the optimal test $T^*$ for at least the first observation. Now suppose the first observation $x_1$ is taken. We then compute the a posteriori probability $g_1$ that $H_1$ is true, where $g_1$ is given by

$$(3.1) \quad g_1 = \frac{g f_1(x_1)}{g f_1(x_1) + (1 - g) f_2(x_1)}.$$

We are now in the same position as we were initially. If $g_1 \leq g$ the best procedure is to stop sampling and accept $H_2$. If $g_1 \geq \tilde{g}$, the best procedure is to stop sampling and accept $H_1$. However if $g < g_1 < \tilde{g}$ then there exists a sequential test $T'$ and hence $T^*$ which is worth performing and we take another observation.

We thus see that the optimum sequential test $T^*$ for a dichotomy must coincide with the following procedure. For any $w_{12}$ and $w_{21}$ we determine $g$ and $\tilde{g}$ by a method to be described later. Let $n = 0, 1, 2, \ldots$, represent the number of observations taken in sequence. At each stage we compute $g_n$ where $g_n$ is given by

$$(3.2) \quad g_n = \frac{g P_{1n}}{g P_{1n} + (1 - g) P_{2n}}.$$

We continue taking observations as long as $g < g_n < \tilde{g}$. We stop as soon as, for some $n$, either $g_n \leq g$ or $g_n \geq \tilde{g}$. In the former case we accept $H_2$, in the latter case we accept $H_1$.

The optimum test $T^*$ described above is identical with the sequential probability ratio test developed by Wald [2]. Wald's test is defined as follows: Let $L_n = P_{2n}/P_{1n}$ and let $A$ and $B$ be two positive numbers with $B \leq 1$ and $A \geq 1$. Observations are taken in sequence and sampling continues as long as $B < L_n < A$. Sampling terminates as soon as for some sample size $n$, either $L_n \geq A$ or $L_n \leq B$. In the former case, $H_2$ is accepted, and in the latter case $H_1$ is accepted. This procedure, however, is the same as $T^*$ provided $T^*$ requires at least one observation and provided we set

$$(3.3) \quad B = \frac{1 - \tilde{g}}{\tilde{g}} \frac{g}{1 - g} \quad \text{and} \quad A = \frac{1 - g}{g} \frac{g}{1 - g}.$$
We now define

\[ z_i = \log \frac{f_2(x_i)}{f_1(x_i)}, \]

\[ a = \log A, \quad -b = \log B. \]

In terms of these quantities, the sequential procedure \( T^* \) can be defined as follows: Continue sampling as long as \(-b < \sum_{i=1}^n z_i < a\). Terminate sampling and accept the appropriate hypothesis as soon as for some \( n, \sum_{i=1}^n z_i > a \) or \( \sum_{i=1}^n z_i < -b \).

A Method for Determining \( g \) and \( \bar{g} \).

From the above considerations we see that the optimum sequential test for a dichotomy with a constant cost function is completely determined by \( g \) and \( \bar{g} \). We shall therefore turn our attention to the problem of determining these quantities for any given \( w_{12} \) and \( w_{21} \). However, before we consider this problem we state the following theorem which will be proved in Section 6: Let \( \Sigma \) be any class of sequential tests \( T \) for a dichotomy \( H_1, H_2 \). Let \( R(g \mid T) \) be the average risk of test \( T \) when the a priori probability of \( H_1 \) is \( g \). Then \( \inf_{T \in \Sigma} R(g \mid T) \) is a continuous function of \( g \) in the open interval \( 0 < g < 1 \).

The above theorem implies that the risk of the sequential test \( T^* \) which is best among the class of tests involving taking at least one observation is a continuous function of \( g \). Hence it follows that at the boundaries \( g = g \) as well as \( g = \bar{g} \), the risk incurred when the appropriate decision is made with no observations must equal the average risk when observations are taken and the optimal procedure is used thereafter. Thus if we equate these two risks at \( g = g \) and also at \( g = \bar{g} \), we get two equations from which we can obtain these quantities provided, of course, we are able to compute the operating characteristics of the sequential probability ratio test. Since the two risks are equal at \( g = g \) and at \( g = \bar{g} \), it follows, as previously noted, that it makes no difference whether or not those points are included in the intervals \( I_1 \) and \( I_2 \), respectively, where no further observations are taken.

When \( g = g \), the risk of accepting \( H_2 \) with no observations is given by

\[ R_2 = w_{12} g. \]

On the other hand, if we set \( g = \bar{g} \) in (3.5) we obtain a sequential probability ratio test \( T^*_\bar{g} \) defined by the boundaries

\[ g = \log A = 0 \quad \text{and} \quad -b = \log B = \log \frac{1 - \bar{g}}{\bar{g}} \frac{g}{1 - g} \]
We are therefore led to the problem of determining the average risk \(R(g \mid T^*_q)\) where \(T^*_q\) is defined by (3.7).

It is clear that for this test, the first observation will either terminate the sampling or result in a new sequential test with boundaries \(a' = -z_1\) and \(-b' = -(b + z_1)\) where \(z_1\) is given by (3.4) with \(i = 1, 2\), and \(-b < z_1 < 0\). Let \(P_i(z)\) be the probability distribution of \(z = \log f_2(x)/f_1(x)\) when \(H_i\) is true, \((i = 1, 2)\). Moreover, for any sequential probability ratio test defined by boundaries \(-b'\) and \(a'\), let \(L(a', b' \mid H_i) = P(\sum_{j=1}^n z_j \leq -b')\) when \(H_i\) is true. Then \(1 - L(a', b' \mid H_i) = P(\sum_{j=1}^n z_j \geq a')\) when \(H_i\) is true. We also define \(\bar{c}(n \mid a', b'; H_i)\) as the average number of observations required to reach a decision with this test when \(H_i\) is true.

Under the hypothesis \(H_1\), the average risk when \(T^*_q\) is used, is given by

\[
R_1(T^*_q) = 1 + w_{12} \int_0^\infty dP_1(z) + \int_{-b}^0 \bar{c}(n \mid -z, b + z; H_1) dP_1(z)
\]

(3.8)

Under the hypothesis \(H_2\), the average risk with \(T^*_q\) is given by

\[
R_2(T^*_q) = 1 + w_{21} \int_{-\infty}^{-b} dP_2(z) + \int_{-b}^0 \bar{c}(n \mid -z, b + z; H_2) dP_2(z)
\]

(3.9)

Hence the total risk using \(T^*_q\) when \(g = g\) is

\[
R(g \mid T^*_q) = gR_1(T^*_q) + (1 - g)R_2(T^*_q).
\]

(3.10)

Equating (3.10) to (3.6) we get

\[
gR_1(T^*_q) + (1 - g)R_2(T^*_q) = w_{12}g.
\]

(3.11)

By symmetry, we get

\[
\bar{g}R_1(T^*_q) + (1 - \bar{g})R_2(T^*_q) = w_{21}(1 - \bar{g}).
\]

(3.12)

Equations (3.11) and (3.12) determine \(g\) and \(\bar{g}\) uniquely. In general, it will be very difficult to compute the operating characteristics involved in these equations. However, it is possible to employ the approximations developed by Wald [2] which will usually result in values of \(g\) and \(\bar{g}\) close to the true values, especially if the hypotheses \(H_1\) and \(H_2\) do not differ much from each other.
Exact Values of \( g \) and \( \tilde{g} \) for a Special Class of Double Dichotomies

We are given two binomial populations \( \pi_1 \) and \( \pi_2 \) defined by two parameters \( p_i \) and \( q_i \) where \( p_i = P(x = 1) \) and \( q_i = 1 - p_i = P(x = 0) \) for \( i = 1, 2 \) and \( p_1 < p_2 \). Let \( H_1 \) stand for the hypothesis that \( p_1 \) is associated with \( \pi_1 \) and \( p_2 \) is associated with \( \pi_2 \), and let \( H_2 \) stand for the hypothesis that \( p_2 \) is associated with \( \pi_1 \) and \( p_1 \) is associated with \( \pi_2 \). Let \( \psi_{12} \) be the risk of accepting \( H_2 \) when \( H_1 \) is true and \( \psi_{21} \) be the risk of accepting \( H_1 \) when \( H_2 \) is true. We assume that the cost per observation is constant and, without loss of generality, is taken as unity. Let \( g \) be the a priori probability that \( H_1 \) is true and hence \( 1 - g \) is the a priori probability that \( H_2 \) is true. The problem is to determine \( g \) and \( \tilde{g} \) which define the optimum procedure \( T^* \) for testing these hypotheses.

It is easily shown that if \( g \) and \( \tilde{g} \) are known, \( T^* \) is defined as follows: We set

\[
(3.13) \quad a = \left\{ \frac{\log \frac{1 - \tilde{g}}{\tilde{g}} - g}{1 - g} \right\}, \quad b = \left\{ \frac{\log \frac{1 - g}{\tilde{g}} - \frac{g}{1 - g}}{\log \frac{p_2 q_1}{q_2 p_1}} \right\},
\]

where the symbol \( \{y\} \) stands for the smallest integer greater than or equal to \( y \). Let \( x_{11}, x_{12}, \ldots \), be a sequence of observations obtained from \( \pi_1 \) and \( x_{21}, x_{22}, \ldots \), a sequence of observations obtained from \( \pi_2 \). We continue sampling as long as \(-b < \sum_{i=1}^{n} (x_{2i} - x_{1i}) < a\). We terminate sampling as soon as for some sample size \( n \) either \( \sum_{i=1}^{n} (x_{2i} - x_{1i}) = a \) or \( \sum_{i=1}^{n} (x_{2i} - x_{1i}) = -b \). In the former case we accept \( H_1 \), in the latter case we accept \( H_2 \).

Let \( L(a, b | H_i) = P \left[ \sum_{i=1}^{n} (x_{2i} - x_{1i}) = a \right] \) when \( H_i \) is true and let \( \mathcal{E}(n | a, b; H_i) \) be the expected number of observations required to reach a decision when \( H_i \) is true. Then, without any approximation (see [3]) we have

\[
(3.14) \quad L(a, b | H_1) = \frac{u^{a+b} - u^b}{u^{a+b} - 1},
\]

\[
(3.15) \quad L(a, b | H_2) = \frac{\left( \frac{1}{u} \right)^{a+b} - \left( \frac{1}{u} \right)^b}{\left( \frac{1}{u} \right)^{a+b} - 1},
\]

\[
(3.16) \quad \mathcal{E}(n | a, b; H_1) = \frac{(a + b) L(a, b | H_1) - b}{p_1 q_2 - p_2 q_1},
\]

\[
(3.17) \quad \mathcal{E}(n | a, b; H_2) = \frac{(a + b) L(a, b | H_2) - a}{p_2 q_1 - p_1 q_2},
\]
(3.18) \[ u = \frac{p_2 q_1}{p_1 q_2}. \]

Now let \( g = g. \) Then \( T^*_g \) is defined by the boundaries \( a = a \) and \( b = 0 \) where \( a \) is obtained from (3.13) with \( g = g. \)

Using the same considerations as on page 227, we find the average risk of \( T^*_g \) to be,

\[ R(g \mid T^*_g) = 1 + gw_{12}(1 - p_2 q_1) \]

\[ + gp_2 q_1 [\xi(n \mid a - 1, 1; H_1) + w_{12}[1 - L(a - 1, 1 \mid H_1)] \]
\[ + (1 - g)p_1 q_2 [\xi(n \mid a - 1, 1; H_2) + w_{21}L(a - 1, 1 \mid H_2)]. \]

If we equate (3.19) to \( w_{12}g \), the risk of accepting \( H_2 \) with no observations, and simplify, we get

\[ 1 + p_1 q_2 \xi(n \mid a - 1, 1; H_2) + p_1 q_2 w_{21}L(a - 1, 1 \mid H_2) \]

\[ + g[p_2 q_1 [\xi(n \mid a - 1, 1; H_1) - w_{12}L(a - 1, 1 \mid H_1)] \]
\[ - p_2 q_2 [\xi(n \mid a - 1, 1; H_2) + w_{21}L(a - 1, 1 \mid H_2)] = 0. \]

If we now let \( g = \bar{g} \), then \( T^*_g \) is defined by the boundaries \( a = 0 \) and \( b = \bar{a} \). Hence the average risk of going on with the optimum procedure when \( g = \bar{g} \) is given by

\[ R(\bar{g} \mid T^*_g) = 1 + \bar{g} p_1 q_2 [\xi(n \mid 1, a - 1; H_1) \]
\[ + w_{12}[1 - L(1, a - 1 \mid H_1)] \]
\[ + (1 - \bar{g})p_1 q_2 [\xi(n \mid 1, a - 1; H_2) + w_{21}L(1, a - 1 \mid H_2)] \]

Equating (3.21) to \( w_{21}(1 - \bar{g}) \) and simplifying, we get

\[ 1 - p_2 q_2 w_{12} + p_2 q_1 \xi(n \mid 1, a - 1; H_2) + p_2 q_1 w_{21}L(1, a - 1 \mid H_2) \]
\[ + \bar{g}[p_1 q_2 \xi(n \mid a - 1, 1; H_1) + w_{12}p_1 q_2 \]
\[ - p_1 q_2 w_{12}L(1, a - 1 \mid H_1) + w_{21}p_2 q_1 \]
\[ - p_2 q_1 \xi(n \mid 1, a - 1; H_2) - w_{21}p_2 q_1 L(1, a - 1 \mid H_2)] = 0. \]

The following procedure may be used for computing \( g \) and \( \bar{g} \) from equations (3.20) and (3.22). Guess an integer \( \bar{a} \) and compute \( g \) from (3.20) and \( \bar{g} \) from (3.22). Substitute these values in the formula

\[ a = \left\{ \begin{array}{ll} \log \frac{\bar{g}(1 - g)}{g(1 - \bar{g})} \\ \log \frac{p_2 q_1}{p_1 q_2} \end{array} \right\}. \]
If the resulting quantity has a value equal to the guessed \( \hat{a} \), the computed \( g \) and \( \hat{g} \) are correct. If not, repeat the process.

Equations (3.20) and (3.22) can also be used to compute \( w_{12} \) and \( w_{21} \) for given values of \( g \) and \( \hat{g} \). This can be accomplished as follows. For the given \( g \) and \( \hat{g} \), compute \( a \) from (3.23). Set \( \hat{a} = a \) in (3.20) and (3.22) and solve for \( w_{12} \) and \( w_{21} \).

The average risk of the optimum sequential procedure under consideration can be computed as a function of \( g \) as soon as \( g \) and \( g^* \) are determined and is given by

\[
R(g \mid T^*) = g\hat{C}(n \mid a, b; H_1) + (1 - g)\hat{C}(n \mid a, b; H_2) \\
+ gw_{12}[1 - L(a, b \mid H_1)] + (1 - g)w_{21}L(a, b \mid H_2)
\]

where for the given \( g \), \( a \) and \( b \) are determined from (3.13). Since \( a \) and \( b \) are integers, the curve obtained by plotting \( R(g \mid T^*) \) against \( g \) will consist of connected line segments.

4. MULTI-VALUED DECISIONS AND THE THEORY OF GAMES

The finite multi-valued decision problem can be considered as a game with Nature playing against the statistician. Nature selects a hypothesis \( H_i (i = 1, 2, \ldots, k) \) and the statistician selects a test procedure. The pay-off function involved in this game is the risk function which consists of the average cost of experimentation plus the average loss incurred in making erroneous decisions under the test procedure selected by the statistician.

From the point of view of the theory of games, the existence of the optimum sequential procedure \( T^* \) for multi-valued decisions means this: For every mixed strategy (a priori distribution) of Nature, the statistician has a pure strategy which is optimum against it. Thus, if Nature's mixed strategy were known, the statistician's problem would be solved, except for the problem of characterizing \( T^* \). But often Nature's mixed strategy is completely unknown. In that case Wald suggests that the statistician play a minimax strategy: that is, the statistician should select that decision procedure which minimizes the maximum risk. This procedure has the property that if consistently employed, the resulting average risk will be independent of Nature's a priori distribution, provided Nature's best strategy is completely mixed.

Examples of Dichotomies

In terms of the multi-valued decision problem, say the dichotomy with a constant cost function, the minimax strategy of the statistician consists of the following: For a given \( H_1, H_2, w_{12}, w_{21} \) and cost per observation \( c \) (which we take as unity) the statistician computes \( g \) and \( \hat{g} \) and then finds that \( g = g^* \) for which the risk function \( R(g^* \mid T^*_\phi) \) is a
maximum. He then proceeds as if Nature always selects \( g^* \) for its a priori distribution. If the hypotheses \( H_1 \) and \( H_2 \) involve continuous random variables, this procedure will always result in an average risk \( R(g \mid T^*) \) which is independent of \( g \). However, if \( H_1 \) and \( H_2 \) involve discrete random variables, this is no longer generally true and the statistician may have to resort to mixed strategies.

As an illustration, consider the double-dichotomy discussed in the preceding section. Suppose we have obtained \( g \) and \( \bar{g} \) and found that \( R(g \mid T^*) \) given in (3.24) has a maximum at \( g = g^* \). Let the \( a \) and \( b \) corresponding to \( g^* \) be designated by \( a^* \) and \( b^* \). Then, in order that the sequential test \( T^*_a \) (defined by the boundaries \( a^* \) and \( -b^* \)) be independent of \( g \), we must have (see 3.24),

\[
\bar{c}(n \mid a^*, b^*; H_1) + w_{12}[1 - L(a^*, b^* \mid H_1)] = \bar{c}(n \mid a^*, b^*; H_2) + w_{21}L(a^*, b^* \mid H_2).
\]

But since \( a^* \) and \( b^* \) are necessarily integers, this equation will usually not be satisfied. This implies that Nature’s strategy \( g^* \) must have the property that when we set \( g = g^* \) in (3.13) (removing the braces) either \( a \) is exactly an integer, or \( b \) is exactly an integer, or both are integers. Suppose for example that \( a = a^* \) is an integer but not \( b = b^* \). This means that when \( \sum_{i=1}^{n} (x_{2i} - x_{1i}) = a^* \) we have two courses of action to follow. We can either stop and accept \( H_1 \) or go on experimenting with the best sequential test defined by the boundaries \( a = a^* + 1 \) and \( b = b^* \). But these two procedures will not always have the same risk for \( g \neq g^* \). Thus in order to make the average risk independent of \( g \) we shall have to employ a mixed strategy. That is, we shall have to employ one procedure some specified fraction \( a \) of the times and the other procedure, \( 1 - a \) of the times, where \( 0 < a < 1 \).

To illustrate these concepts, we have computed four examples of binomial dichotomies all with the same hypotheses \( H_1 : p_1 = 1/3 \), \( H_2 : p_2 = 2/3 \) but varying \( w_{12} \) and \( w_{21} \). The risk function \( R(g \mid T^*) \) as well as the minimax strategies are given for each of these examples\(^6\) in Figures 1 to 4.

It can be shown that for all symmetric binomial dichotomies there always exists a pure minimax strategy for the statistician if \( w_{12} = w_{21} \). However, this is no longer true if \( w_{12} \neq w_{21} \). This phenomenon is illustrated in the charts following.

\(^6\) The optimum sequential test \( T^* \) for any symmetric binomial dichotomy (i.e., \( p_1 + p_2 = 1 \)) becomes identical with test described in Section 5 when the following substitutions are made: write \( q_1 \) for \( p_1 q_2 \), \( q_2 \) for \( p_2 q_1 \), \( \sum_{i=1}^{N} x_i \) for \( \sum_{i=1}^{N} (x_{2i} - x_{1i}) \), where each \( x_i \) takes on the value of 1 with probability \( p_i \) and -1 with probability \( 1 - p_i (i = 1, 2) \).
AVERAGE MINIMAX RISK = 4.3

THE AVERAGE MINIMAL RISK AS A FUNCTION
OF THE A PRIORI DISTRIBUTION $g$ FOR A
DICHOTOMY WITH $H_1: p_1 = \frac{1}{2}$, $H_2: p_2 = \frac{3}{5},$
$C=1, M_0 = M_1 = 10$

MINIMAX STRATEGIES:
FOR NATURE: $\frac{9}{10}, \frac{11}{10}$
FOR THE STATISTICIAN: SEQUENTIAL
PROBABILITY RATIO TEST $(0, 1, 1, 1)$

FIGURE 1

AVERAGE MINIMAX RISK = 3.8

THE AVERAGE MINIMAL RISK AS A FUNCTION
OF THE A PRIORI DISTRIBUTION $g$ FOR A
DICHOTOMY WITH $H_1: p_1 = \frac{1}{2}$, $H_2: p_2 = \frac{3}{5},$
$C=1, M_0 = 0, M_1 = 10$

MINIMAX STRATEGIES:
FOR NATURE: $\frac{9}{10}, \frac{11}{10}$
FOR THE STATISTICIAN: SEQUENTIAL
PROBABILITY RATIO TEST $(0, 1, 1, 1)$
WITH FREQUENCY $\frac{3}{5}$ AND ACCEPTING $H_2$
WITH NO OBSERVATIONS WITH FREQUENCY $\frac{2}{5}$

FIGURE 2
THE AVERAGE MINIMAL RISK AS A FUNCTION OF THE PRIOR DISTRIBUTION $\theta$ FOR A
DICHOTOMY WITH $H_1: \theta = \frac{1}{2}, H_2: \theta = \frac{3}{4}$.
$G(1), \text{in} \{2,5,100\}$.

**Figure 3**

THE AVERAGE MINIMAL RISK AS A FUNCTION OF THE PRIOR DISTRIBUTION $\theta$ FOR A
DICHOTOMY WITH $H_1: \theta = \frac{1}{3}, H_2: \theta = \frac{2}{3}$.
$G(1), \text{in} \{2,5,100\}$.

**Figure 4**
Examples of Trichotomies

Example 1. Assume that the random variables $x_1, x_2, \ldots$ are independently distributed and all have the same distribution. Each $x_n$ takes on only the values 1, 2, 3 with probabilities specified by one of the following alternative hypotheses:

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>1 2 3</td>
</tr>
<tr>
<td>$H_2$</td>
<td>0 $\frac{1}{2}$ 0</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$\frac{1}{2}$ 0</td>
</tr>
</tbody>
</table>

Let $w_{ij}$ be the loss if $H_j$ is accepted when $H_i$ is true; the values are given by the following table:

<table>
<thead>
<tr>
<th>State of Nature</th>
<th>Hypothesis Accepted</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>$H_2$   $H_3$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>0        4 6</td>
</tr>
<tr>
<td>$H_2$</td>
<td>6        0 4</td>
</tr>
<tr>
<td>$H_3$</td>
<td>4        6 0</td>
</tr>
</tbody>
</table>

Note that both of these matrices are invariant under a cyclic permutation of the hypotheses and events. Finally, assume that the cost of each observation is 1.

Let $g_i$ be the a priori probability of $H_i$. An a priori distribution $\bar{g} = (g_1, g_2, g_3)$, with $g_1 + g_2 + g_3 = 1$, may be represented by a point in an equilateral triangle with unit altitudes; the distances from the point to the three sides are the values of $g_1$, $g_2$, and $g_3$. $P_i$ is the point where $g_i = 1 (i = 1, 2, 3)$.

Let $R(\bar{g} \mid T)$ be the average risk under sequential procedure $T$ when the a priori probabilities are $g_1, g_2, g_3$. Let $T_0$ be the best sequential procedure where no observations are taken; let $S_i$ be the region in $g$-space where $H_i$ is accepted under $T_0$. Let $L_i(\bar{g})$ be the loss in accepting $H_j$ when the a priori distribution is $\bar{g}$. Then

$$L_i(\bar{g}) = \sum_{i=1}^{3} w_{ij} g_i.$$  (4.2)

$S_i'$ is defined by the inequalities

$$L_1 \leq L_2, \quad L_1 \leq L_3,$$

or

$$6g_2 + 4g_3 \leq 4g_1 + 6g_3,$$

$$6g_2 + 4g_3 \leq 6g_1 + 4g_2.$$
That is,

\[ g_1 \geq \max (\frac{3}{2}g_2 - \frac{1}{2}g_3, \frac{1}{2}g_2 + \frac{3}{2}g_3). \]

At \( P_1 \), \( g_2 = g_3 = 0 \), \( g_1 = 1 \), so that \( P_1 \) belongs to \( S'_1 \). When \( g_3 = 0 \), (4.3) becomes \( g_1 \geq \frac{3}{2}g_2 \), while \( g_1 + g_2 = 1 \), so that \( g_1 \geq \frac{3}{5} \); when \( g_2 = 0 \), \( g_1 \geq \frac{2}{3}g_3 \), \( g_1 + g_3 = 1 \), so that \( g_1 \geq \frac{2}{5} \). Also, the two lower bounding lines for \( g_1 \) are equal when \( \frac{3}{2}g_2 - \frac{1}{2}g_3 = \frac{1}{3}g_2 + \frac{2}{3}g_3 = g_1 \), or \( g_1 = g_2 = g_3 = \frac{1}{3} \). \( S'_1 \) contains all points above the boundary defined by the polygon with vertices \( (\frac{3}{5}, \frac{2}{5}, 0), (\frac{1}{3}, \frac{1}{3}, 1/3), \) and \( (\frac{2}{5}, 0, \frac{3}{5}) \). \( S'_2 \) and \( S'_3 \) can be obtained by successive cyclic permutations of these coordinates.

\[ \text{Figures 5 and 6} \]

Let \( T^*(j) \) be the optimum sequential procedure for a given a priori distribution. Let \( g_{ij} \) be the a posteriori probability of \( H_i \) given that \( x_1 = j \);

\[ g_{ij} = \frac{g_i p_{ij}}{\sum_{k=1}^3 g_k p_{kj}} \]

where \( p_{ij} \) is the probability that \( x_1 = j \) under \( H_i \). Let \( T^*_1(\tilde{g}) \) be the sequential procedure defined as taking one observation and then using procedure \( T^*(g_{1j}, g_{2j}, g_{3j}) \) when \( x_1 = j \). \( T^*_1(\tilde{g}) \) is the best sequential procedure which involves taking at least one observation. In the present case, \( p_{ij} = 0 \), so that \( g_{ij} = 0 \) by (4.4); therefore, \( T^*(g_{1j}, g_{2j}, g_{3j}) \) is the optimum sequential test for a dichotomy. \(^7\)

Let \( S^*_i \) be the region in \( g \)-space for which \( H_i \) is accepted without any observation under \( T^*(\tilde{g}) \). As has been shown, the regions \( S^*_i \) are essen-

\(^7\) It is to be pointed out that, for any trichotomy, the intersection of the regions \( S^*_i \) with the sides of the triangle may be determined by computing \( g, \tilde{g} \) [see Section 3] for the appropriate dichotomy.
tially all that is needed to determine all the tests $T^*(\bar{g})$. Further, the regions $S_i^*$ are convex sets whose boundaries are characterized by the relations

$$R[\bar{g} \mid T_1(g)] = L_i(g), \quad L_i(g) = \min_i L_i(g),$$

so that

$$S_i^* \subset S'_i.$$  

It is first necessary to find the optimum tests for each of the dichotomies formed by taking pairs from the trichotomy $H_1, H_2, H_3$. Consider the dichotomy $H_1, H_2$. Then $g_1 + g_2 = 1$. Suppose $g_1$ is such that it pays to take at least one observation. From (4.4), since $p_{i3} = 1/2(i = 1, 2)$,

$$g_{i3} = \frac{g_i}{g_1 + g_2} = g_i.$$  

Hence, if $x_1 = 3$, the a posteriori probabilities are unchanged, and, as shown previously, the optimum test calls for taking another observation. On the other hand, $p_{11} = 0$, so that $g_{21} = 1$. Therefore, if $x_1 = 1$, the process should be terminated and $H_2$ accepted. Similarly, if $x_1 = 2$, the process should be terminated and $H_1$ accepted. It follows that if $g_1$ is such that the best sequential procedure calls for taking at least one observation, then the best procedure is to sample indefinitely until $x_n = 1$ or $2$; in the former case, accept $H_2$, in the latter, $H_1$. The probability of accepting the wrong hypothesis is zero under either hypothesis; the risk is then the expected number of observations, which is 2 under either hypothesis.

The boundaries $g_{12}$ and $g_{12}$ of the interval of $g_1$'s in which it pays to go on are then determined by the equation,

$$w_{12}g_{12} = 2, \text{ or } g_{12} = \frac{1}{2},$$  

$$w_{21}(1 - g_{12}) = 2 \text{ or } g_{12} = \frac{3}{2}.$$  

Returning to the specification of $T_1^*(\bar{g})$, we note that if $x_1 = 3$,

$$g_{i3} = \frac{g_i p_{i3}}{\sum_k g_k p_{k3}}.$$  

As $p_{33} = 0$, $p_{i3} = 1/2$ for $i = 1, 2$,

$$g_{i2} = \frac{g_i}{g_1 + g_2} (i = 1, 2).$$
Therefore, $T^*_1(j)$ can be described as follows: If $x_i = 3$, $H_3$ is rejected entirely; if $g_i/(g_1 + g_2) \leq 1/2$, stop and accept $H_2$; if $g_i/(g_1 + g_2) \geq 2/3$, stop and accept $H_1$; if $1/2 < g_i/(g_1 + g_2) < 2/3$, continue sampling until $x_n = 1$ or $2$, at which point stop and accept $H_2$ or $H_1$, respectively. The three conditions on $g_i/(g_1 + g_2)$ can be written in the simpler form,

$$g_1 \leq g_2, \quad g_1 \geq 2g_2, \quad g_2 < g_1 < 2g_2,$$

respectively. The cases where $x_1 = 1$ or $2$ can be obtained from the preceding case by cyclic permutation of the numbers $1, 2, 3$.

Let $R^*_i(j)$ be the conditional expected risk associated with $T^*_1(j)$ when the a priori probability distribution is $\tilde{g}$, given that $x_i = j$; and let $p_i$ be the a priori probability that $x_i = j$. Then

$$R[\tilde{g} \mid T^*_1(j)] = \sum_{i=1}^{3} p_i R^*_i(j).$$

$$p_i = \sum_{i=1}^{3} p_{ij} g_i.$$

R\_1^*(\tilde{g}) = \begin{cases} 1 + \frac{w_{12} g_1}{g_1 + g_2} = 1 + \frac{4g_1}{g_1 + g_2} & \text{if } g_1 \leq g_2, \\
3 & \text{if } g_2 \leq g_1 \leq 2g_2, \\
1 + \frac{w_{21} (1 - \frac{g_1}{g_1 + g_2})}{g_1 + g_2} = 1 + \frac{6g_2}{g_1 + g_2} & \text{if } g_1 \geq 2g_2. \end{cases}$$

$R^*_1(\tilde{g})$ and $R^*_2(\tilde{g})$ can be obtained from (4.12) by cyclic permutation of the subscripts.

The region $S^*_1$ can now be determined by the relations (4.5–6) and (4.10–12). First note that when $g_3 = 0$, the problem reduces to the dichotomy between $H_1$ and $H_2$ already discussed, so that the interval from $g_1 = 1$ to $g_1 = 2/3$ on the line $g_3 = 0$ belongs to $S^*_1$. Hence the point $(2/3, 1/3, 0)$ lies on the boundary of $S^*_1$. This point satisfies the conditions,

$$g_2 \leq g_1 \leq 2g_2, \quad g_2 \geq 2g_2, \quad g_2 \leq g_1.$$

Consider the intersection, if any, of the boundary of $S^*_1$ with the region $R_1$ defined by (4.13). Using (4.10–11), (4.5),

$$\frac{g_1 + g_2}{2} R^*_3(\tilde{g}) + \frac{g_2 + g_3}{2} R^*_1(\tilde{g}) + \frac{g_1 + g_3}{2} R^*_2(\tilde{g}) = L_1(\tilde{g}).$$

From (4.13–14), (4.12), and (4.2),
\[ 3 \frac{g_1 + g_2}{2} + \frac{g_2 + g_3}{2} \left( 1 + \frac{6g_3}{g_2 + g_3} \right) + \frac{g_1 + g_2}{2} \left( 1 + \frac{4g_3}{g_1 + g_3} \right) = 6g_2 + 4g_3, \]

or

(4.15) \[ g_2 = \frac{1}{3}. \]

The intersection of (4.15) with the line \( g_2 = 2g_3 \) occurs at the point \((1/2, 1/3, 1/6)\), which satisfies the conditions (4.13) and so lies on the boundary of the region \( R_1 \). As \( R_1 \) is convex, it follows that the boundary of \( S_1^* \) actually does intersect \( R_1 \) and there coincides with the line segment joining \((2/3, 1/3, 0)\) and \((1/2, 1/3, 1/6)\). The latter point satisfies also the conditions (4.16)

\[ g_2 \leq g_1 \leq 2g_2, \quad g_3 \leq g_2 \leq 2g_3, \quad g_3 \leq g_1. \]

Let \( R_2 \) be the region defined by (4.16). Then we can find as before the intersection of the boundary of \( S_1^* \) with \( R_2 \); the boundary hits the line \( g_2 = g_3 \) at the point \((3/7, 2/7, 2/7)\), which point lies in \( R_2 \). Hence the boundary of \( S_1^* \) actually does intersect \( R_2 \) and there coincides with the segment joining \((1/2, 1/3, 1/6)\) to \((3/7, 2/7, 2/7)\). If we continue this method, it can be shown that \( S_1^* \) is bounded by the polygon with vertices \((2/3, 1/3, 0)\), \((1/2, 1/3, 1/6)\), \((3/7, 2/7, 2/7)\), \((2/5, 1/5, 2/5)\), \((1/2, 0, 1/2)\), and \((1, 0, 0)\). It is easily verified that \( S_1^* \) is actually a subset of \( S_1^* \), as demanded by (4.6). The vertices of the polygons bounding \( S_2^* \) and \( S_3^* \) can be obtained by cyclic permutation of the coordinates.

For any given \( \tilde{g} \), the regions \( S_1^* \), \( S_2^* \), and \( S_3^* \) completely define the optimal procedure. It remains to find the minimax procedure.

As shown by (4.12), the maximum conditional expected risk given that \( x_1 = 1 \) is 3, and this occurs when \( g_3 \leq g_2 \leq 2g_3 \). Similarly, the maximum conditional expected risks given that \( x_1 = 2 \) and \( 3 \), respectively, are both equal to 3, and they occur when \( g_1 \leq g_3 \leq 2g_1 \), \( g_2 \leq g_1 \leq 2g_2 \), respectively. Any \( \tilde{g}^* \) satisfying these three conditions will be a least favorable a priori distribution; clearly, the only set of values is \( g_1 = g_2 = g_3 = 1/3 \). If \( x_1 = 1 \), the corresponding a posteriori distribution is \((0, 1/2, 1/2)\). This is on the boundary of \( S_2^* \), so that the optimum procedure is to stop after one observation and choose \( H_3 \). In general, then, the minimax procedure is to take one observation, stop, and accept \( H_2 \) if \( x_1 = 1, H_1 \) if \( x_1 = 2 \), and \( H_2 \) if \( x = 3 \). The risk associated with this test is 3, of which the cost of observation is 1, and the expected loss due to incorrect decision is 2, independent of the true a priori distribution.

It may be of interest to note that the minimax test is not unique, the
lack of uniqueness corresponding exactly to the inclusion or exclusion of the boundaries of \( S^*_1, S^*_2, \) and \( S^*_3 \) in those sets. If we exclude the boundaries, then, when \( x_1 = 1 \), we continue. So long as \( x_n = 1 \), the a posteriori probabilities remain at \((0, 1/2, 1/2)\); when \( x_n = 2 \) \((3)\), the a posteriori probability of \( H_3(H_2) \) becomes 1. Therefore, a second minimax test is to stop the first time \( x_n \neq x_{n-1} \), and then accept that hypothesis whose subscript equals neither \( x_n \) nor \( x_{n-1} \). The maximum risk is again 3; all of this is represented by the expected number of observations which is the same for all a priori distributions.

**Example 2.** The boundaries of the regions \( S^*_i \) might also be straight lines, as shown by the following example: Let \( x_1, x_2, \ldots \), be independently distributed with the same distribution, and \( x_n \) takes on the values 1, 2, or 3. Let \( H_i \) be the hypothesis that \( x_i = i \) with probability 1 \((i = 1, 2, 3)\), and let \( w_{ij} = 3 \) for \( i \neq j \), \( w_{ii} = 0 \). Assume the cost of each observation is 1. Then the best test involving at least one observation is clearly to take exactly one observation and accept \( H_i \) if \( x = i \). The expected loss due to incorrect decision is 0, so that the risk of this test is 1. Hence the boundary of \( S^*_1 \) is characterized by the relation, \( w_{21}g_2 + w_{31}g_3 = 3(g_2 + g_3) = 1 \), or \( g_1 = 2/3 \). Similarly, \( S^*_2, S^*_3 \) are defined by the inequalities \( g_2 \geq 2/3, g_3 \geq 2/3 \), respectively.
If no observations are taken, the region $S'_1$ in which $H_1$ is accepted is characterized by

$$
W_{21} + W_{31} \leq \min (W_{11} + W_{23}, W_{12} + W_{32}) \quad \text{or} \quad 1 - g_1 \leq \min (g_1 + g_3, g_1 + g_2),
$$

$$
2g_1 \geq \max (1 - g_2, 1 - g_3).
$$

The boundary is the polygonal line with vertices $(1/2, 1/2, 0)$, $(1/3, 1/3, 1/3)$, and $(1/2, 0, 1/2)$. The boundaries of $S'_2$ and $S'_3$ are found similarly. The regions $S'_1$, $S'_2$, $S'_3$ clearly lie inside $S_1$, $S_2$, $S_3$, respectively.

**EXAMPLE 3.** In both previous examples, inner boundaries of the regions $S'_i$ were found by equating the risk of accepting $H_i$ if no observations are taken with the risk under the best procedure calling for taking at least one observation. The regions so found were in both cases subsets of $S'_i$. However, this relation need not hold in general, as shown by the following example:

Let all conditions be the same as in Example 2 except that $w_{ij} = 5/5$ for $i \neq j$. Then the risk of accepting $H_1$ without observations is equal to the risk of the best test taking at least one observation when $g_1 = 2/5$. But the region $g_1 \geq 2/5$ is not a subset of $S'_1$ (which is the same here as in Example 2). $S'_1$ is the intersection of $S'_1$ and the region $g_1 \geq 2/5$. $S'_1$ is bound from below by the polygonal line with vertices $(1/2, 1/2, 0)$, $(2/5, 2/5, 1/5)$, $(2/5, 1/5, 2/5)$, and $(1/2, 0, 1/2)$.

5. ANOTHER OPTIMUM PROPERTY OF THE SEQUENTIAL PROBABILITY RATIO TEST

In section 3 we have shown that the sequential probability ratio test is optimum in the sense that for a given a priori distribution $q$ it minimizes the average risk. We shall now prove that for fixed probability of making erroneous decisions, this test minimizes the average number of observations when $H_1$ is true as well as when $H_2$ is true.
PROOF: For a fixed $A \geq 1$ and $B \leq 1$, let $\alpha$ be the probability of accepting $H_2$ when $H_1$ is true if the sequential probability ratio test $T^*$ with these boundaries is used. Similarly, let $\beta$ be the probability of accepting $H_1$ when $H_2$ is true. The quantities $\alpha$ and $\beta$ are uniquely determined by $A$ and $B$.

Choose any $g$ such that $0 < g < 1$, solve for $g$ and $\hat{g}$ from (3.3) and then compute $w_{12} = w_{12}(g, \hat{g})$ and $w_{21} = w_{21}(g, \hat{g})$ from (3.11) and (3.12). The quantity $\hat{b}$ entering in (3.11) and (3.12) is given by $\log (A/B)$.

The three quantities $w_{12}(g, \hat{g})$, $w_{21}(g, \hat{g})$ and $g$ have the property that if the a priori distribution is $g$ and if the risk of accepting $H_2$ when $H_1$ is true is $w_{12}(g, \hat{g})$ and the risk of accepting $H_1$ when $H_2$ is true is $w_{21}(g, \hat{g})$, then the sequential test $T^*$ has minimum average risk. The average risk under $T^*$ is given by

$$R(g \mid T^*) = g \mathcal{E}(n \mid T^*; H_1)$$

(5.1)

$$+ (1 - g) \mathcal{E}(n \mid T^*; H_2)$$

$$+ g w_{12}(g, \hat{g}) \alpha + (1 - g) w_{21}(g, \hat{g}) \beta.$$  

Now let $T$ be any other test procedure which results in probabilities $\alpha' \leq \alpha$ and $\beta' \leq \beta$ of making erroneous decisions. Then for the same triplet $g$, $w_{12}(g, \hat{g})$ and $w_{21}(g, \hat{g})$ the average risk under $T$ is given by

$$R(g \mid T) = g \mathcal{E}(n \mid T; H_1)$$

(5.2)

$$+ (1 - g) \mathcal{E}(n \mid T; H_2)$$

$$+ g w_{12}(g, \hat{g}) \alpha' + (1 - g) w_{21}(g, \hat{g}) \beta'.$$

Now since $R(g \mid T^*) \leq R(g \mid T)$ we must have

$$g \mathcal{E}(n \mid T^*; H_1) + (1 - g) \mathcal{E}(n \mid T^*; H_2)$$

(5.3)

$$\leq g \mathcal{E}(n \mid T; H_1) + (1 - g) \mathcal{E}(n \mid T; H_2).$$

But the inequality (5.3) must hold for all values of $g$, $0 < g < 1$. Hence from continuity considerations we must have

$$\mathcal{E}(n \mid T^*; H_1) \leq \mathcal{E}(n \mid T; H_1),$$

(5.4)

and

$$\mathcal{E}(n \mid T^*; H_2) \leq \mathcal{E}(n \mid T; H_2).$$

(5.5)

This proves the theorem.
6. CONTINUITY OF THE RISK FUNCTION OF THE OPTIMUM TEST

THEOREM: Let $\Sigma$ be any class of sequential tests of a multiple decision involving a finite number of alternative hypotheses, and let $R(\theta \mid T)$ be the risk of test $T$ when the a priori distribution of the hypotheses is $\theta$. Then $\inf R(\theta \mid T)$ is a continuous function of $\theta$ in the region for which $g_{ij} > 0$ for all $i$, provided that for some $T_0$ in $\Sigma$, $R(\theta \mid T_0)$ is everywhere finite.

PROOF: For each hypothesis $H_i$ and each test procedure $T$ there is a nonnegative risk which is the sum of the expected cost of observations and the expected loss due to failure to make the best decision; call this risk $A_i(T)$. Then

$$ R(\theta \mid T) = \sum_i g_{ij} A_i(T). $$

Let $\theta_0$ be any a priori distribution for which $g_{0i} > 0$ for all $i$, and choose $0 < \delta_0 < \min_i g_{0i}$. Let $G$ be the region in $\theta$-space for which $\sum_i |g_i - g_{0i}| \leq \delta_0$; then $g_{ij} > 0$ for all $i$ and $\theta$ in the compact set $G$. Let $T_0$ be the test referred to in the hypothesis.

$$ \sup_{\theta \in G} \inf_{T \in \Sigma} R(\theta \mid T) = \sup_{\theta \in G} R(\theta \mid T_0) = K < +\infty, $$

the last inequality following since $R(\theta \mid T_0)$ is linear and hence continuous on the compact set $G$.

Let $\Sigma'$ be the subclass of $\Sigma$ for which $\inf_{T \in \Sigma} R(\theta \mid T) \leq K + 1$. As $\Sigma'$ is a subset of $\Sigma$, $\inf_{T \in \Sigma'} R(\theta \mid T) \leq \inf_{T \in \Sigma} R(\theta \mid T)$. Suppose for some $\theta'$ in $G$,

$$ \inf_{T \in \Sigma'} R(\theta' \mid T) < \inf_{T \in \Sigma} R(\theta' \mid T). $$

Then

$$ \inf_{T \in \Sigma} R(\theta' \mid T) = \inf_{T \in \Sigma - \Sigma'} R(\theta' \mid T). $$

But if $T$ belongs to $\Sigma - \Sigma'$,

$$ R(\theta' \mid T) \geq \inf_{\theta \in G} R(\theta \mid T) > K + 1. $$

From (6.4) and (6.2),

$$ \inf_{T \in \Sigma - \Sigma'} R(\theta' \mid T) \geq K + 1 \geq \sup_{\theta \in G} \inf_{T \in \Sigma} R(\theta \mid T) \geq \inf_{T \in \Sigma} R(\theta' \mid T), $$

contradicting (6.3). Hence,

The essential features of the proof of this theorem are due to George Brown.
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(6.5) \( \inf_{T \in \Sigma'} R(\tilde{g} \mid T) = \inf_{T \in \Sigma} R(\tilde{g} \mid T) \)
everywhere in \( G \), and it suffices to consider only tests in \( \Sigma' \).

\( R(\tilde{g} \mid T) \) assumes its minimum in \( G \). Hence,

\[ \inf_{g \in G} R(g \mid T) = R(g'' \mid T) = \sum_i g_i'' A_i(T). \]

As \( g_i'' > 0 \), \( A_i(T) \geq 0 \) for all \( i \),

(6.6) \[ g_i'' A_i(T) \leq R(\tilde{g}'' \mid T) = \inf_{\tilde{g} \in G} R(\tilde{g} \mid T) \leq K + 1, \]
since \( T \) belongs to \( \Sigma' \). Although \( g_i'' \) may vary with \( T \), it must have a positive lower bound because of the compactness of \( G \) and the fact that \( g_i > 0 \) for all \( g \) in \( G \). As \( i \) takes on only a finite number of values, \( g_i'' \) has a positive uniform lower bound. Then (6.6) implies that \( A_i(T) \) is bounded from above uniformly in \( i \) and \( T \). Let \( C \) be this upper bound.

Choose any \( \delta < \delta_0 \), and any \( \tilde{g} \) such that \( \sum_i |g_i - g_{0i}| < \delta \).

\[ |R(\tilde{g} \mid T) - R(\tilde{g}_0 \mid T)| < C\delta, \]
\[ R(\tilde{g} \mid T) - R(\tilde{g}_0 \mid T) < C\delta. \]
\[ \inf_{T \in \Sigma'} R(\tilde{g} \mid T) \leq \inf_{T \in \Sigma'} R(\tilde{g}_0 \mid T) + C\delta. \]

Similarly, \( \inf_{T \in \Sigma'} R(\tilde{g}_0 \mid T) \leq \inf_{T \in \Sigma'} R(\tilde{g} \mid T) + C\delta \), so that \( \inf_{T \in \Sigma'} R(\tilde{g} \mid T) \), and therefore, by (6.5), \( \inf_{T \in \Sigma'} R(\tilde{g} \mid T) \), is continuous at \( g_0 \).

The continuity of \( \inf_{T \in \Sigma'} R(\tilde{g} \mid T) \) does not extend in general to the boundary of \( g \)-space where \( g_i = 0 \) for some \( i \), as is shown by the following example:

Let \( x_1, x_2, \ldots \), be independently distributed variates with a common distribution, each \( x_n \) taking on only the values 1, 2, 3. Let \( H_1 \) be the hypothesis, \( P(x_n = 1) = p_1, P(x_n = 2) = 1 - p_1, P(x_n = 3) = 0 \), \( H_2 \) the hypothesis, \( P(x_n = 1) = p_2, P(x_n = 2) = 1 - p_2, P(x_n = 3) = 0 \), \( H_3 \) the hypothesis \( P(x_n = 1) = P(x_n = 2) = 0, P(x_n = 3) = 1 \). Let the cost of each observation be 1.

Let \( T_0 \) be the minimax test of the dichotomy \( H_1, H_2 \). Let \( c = \sum_{n=1}^{\infty} 1/n^2 \) Let \( q_n \) be defined inductively, as follows:

\[ q_1 = \frac{1}{c}, \quad q_n = \frac{1}{cn^2 \prod_{i=1}^{n-1} (1 - q_i)}. \]

Then test \( T_1 \) is defined as follows: if \( x_n = 1 \) or 2, and the process has not
stopped before \( n \), form the sequence \( y_1, \ldots, y_m \) consisting of all those elements of the sequence \( x_1, \ldots, x_n \) for which \( x_k \neq 3 \). Then either go on or stop and make a decision in accordance with \( T_0 \) applied to the sequence \( y_1, \ldots, y_m \). If \( x_n = 3 \), stop and accept \( H_3 \) with probability \( q_n \), go on with probability \( 1 - q_n \).

\( T_0 \) has a certain risk \( R \) for all \( \overline{g} \) such that \( g_1 + g_2 = 1 \). Choose \( N > R \).

Let \( T_2 \) be the test consisting of taking \( N \) observation and then making the best decision.

Clearly, under \( H_1 \) or \( H_2 \), \( x_n \) is never equal to 3, so that \( y_1, \ldots, y_m \) is the same as \( x_1, \ldots, x_n \), and \( T_1 \) coincides with \( T_0 \). The expected loss for \( T_1 \) under \( H_1 \) or \( H_2 \) is thus \( R \). Under \( H_3 \), \( x_n = 3 \) for all \( n \); hence, the probability of stopping at \( m \) is \( 1/cm^2 \), so that the probability of stopping is 1 but the expected cost and therefore the expected risk is infinite. Therefore, if \( g_3 = 0 \), \( R(\overline{g} | T_1) = R \); but if \( g_3 > 0 \), \( R(\overline{g} | T_1) = + \infty \).

On the other hand, \( R < N \leq R(\overline{g} | T_2) \leq + \infty \), everywhere.

If \( \Sigma \) contains the two tests \( T_1 \) and \( T_2 \), \( \inf_{\tau \in \Sigma} R(\overline{g} | T) = R(\overline{g} | T_1) = R \)

for \( g_3 = 0 \) but \( \inf_{\tau \in \Sigma} R(\overline{g} | T) = R(\overline{g} | T_2) \geq N > R \) for \( g_3 \neq 0 \).

Hence \( \inf_{\tau \in \Sigma} R(\overline{g} | T) \) is not continuous at any point for which \( g_3 = 0 \).

REFERENCES


