On the nth day of Christmas

On the nth day of Christmas, the singer receives all the gifts that he or she received on the $n-1$st day, plus n more gifts. Letting G_n denote the number of gifts received on day n, we have $G_n = G_{n-1} + n$. The solution to this recurrence is $G_n = 1 + 2 + \cdots + n$, which simplifies to $G_n = n(n+1)/2$.

We can prove by induction that the closed-form formula and the recurrence relation agree. They obviously agree for $n = 1$ and $n = 2$. On the third day of Christmas, the singer receives a partridge in a pear tree, two turtle doves, and three French hens, for a total of six gifts. We can confirm that $G_3 = 6$, as expected.. So assume it's true for all n. Then we'd expect $G_n + 1 = (n+1)(n+2)/2$; whereas, using the inductive hypothesis, we have

$$\begin{aligned} G_{n+1} &= G_n + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{n^2 + 3n + 2}{2} \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

as expected. Hence our G_n formula solves the recurrence relation for all n.

Let the total number of gifts received up to (and including) day n be denoted by $S_n = G_1 + G_2 + \cdots + G_n$. Then

$$\begin{aligned} S_n &= \sum_{i=1}^{n} G_i \\ &= \sum_{i=1}^{n} \frac{i(i+1)}{2} \\ &= (1/2) \sum_{i=1}^{n} i(i+1) \\ &= (1/2) \left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \right) \\ &= (1/2) \left[\sum_{i=1}^{n} i^2 + \frac{n(n+1)}{2} \right] \end{aligned}$$

(You can sanity-check this, for instance, with $n = 3$ to confirm that it gives the correct result.)

How do we get a closed form for $\sum_{i=1}^{n} i^2$? A post on Stack Overflow gives us a helpful assist. We start by observing that

$$\begin{aligned} S &= \sum_{i=1}^{n}[(i+1)^3 - i^3] \\ &= (2^3 - 1^3) + (3^3 - 2^3) + ... + [n^3 - (n-1)^3] + [(n+1)^3 - n^3] \\ &= (n+1)^3 - 1 \end{aligned}$$

At the same time,

$$S = \sum_{i=1}^{n}[(i+1)(i^2 + 2i + 1) - i^3],$$

which after a little algebra yields

$$S = 3\sum_{i=1}^{n} i^2 + (3/2)n^2 + (5/2)n.$$

Putting these two together, we have

$$\begin{aligned} (n+1)^3 - 1 &= 3\sum_{i=1}^{n} i^2 + (3/2)n^2 + (5/2)n \\ 3\sum_{i=1}^{n} i^2 &= (n+1)^3 - 1 - (3/2)n^2 - (5/2)n \\ &= (n+1)(n^2 + 2n + 1) - 1 - (3/2)n^2 - (5/2)n \\ &= n^3 + 2n^2 + n + n^2 + 2n + 1 - 1 - (3/2)n^2 - (5/2)n \\ &= n^3 + (2 + 1 - 3/2)n^2 + (1 + 2 - 5/2)n \\ &= n^3 + (3/2)n^2 + (1/2)n \\ &= (2n^3 + 3n^2 + n)/2 \\ &= n(2n^2 + 3n + 1)/2 \\ &= n(n+1)(2n+1)/2 \\ \sum_{i=1}^{n} i^2 &= n(n+1)(2n+1)/6 \end{aligned}$$

Now let's prove this by induction. This formula is obviously correct for $n = 1$ and $n = 2$. Assume it's true for all n. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= (n+1)^2 + \sum_{i=1}^{n} i^2 \\
&= (n+1)^2 + n(n+1)(2n+1)/6 \\
&= (1/6)(n+1)[6(n+1) + n(2n+1)] \\
&= (1/6)(n+1)[6n + 6 + 2n^2 + n] \\
&= (1/6)(n+1)(2n^2 + 7n + 6)
\end{aligned}$$

while the inductive hypothesis predicts that

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= (1/6)(n+1)(n+2)[2(n+1)+1] \\
&= (1/6)(n+1)[2(n+1)(n+2) + n + 2] \\
&= (1/6)(n+1)[2(n^2 + 3n + 2) + n + 2] \\
&= (1/6)(n+1)(2n^2 + 6n + 4 + n + 2) \\
&= (1/6)(n+1)(2n^2 + 7n + 6)
\end{aligned}$$

Hence we have agreement, and the inductive hypothesis is proved.

Backing up: the total number of gifts received up to and including day n is

$$\begin{aligned}
S_n &= (1/2)\sum_{i=1}^{n+1} i^2 + (1/4)n(n+1) \\
&= (1/2)n(n+1)(2n+1)/6 + (1/4)n(n+1) \\
&= n(n+1)(2n+1)/12 + n(n+1)/4 \\
&= n(n+1)(2n+4)/12
\end{aligned}$$

We can prove this by induction as well. On the first day of Christmas, the singer received one partridge in a pear tree, so $S_1 = 1$. On the second day of Christmas, the singer received the partridge in a pear tree, plus two turtle doves, so $S_2 = 2+1 = 3$. We can confirm that the formula for S_n gives the correct predictions for $n = 1$ and $n = 2$. Now assume it gives correct predictions for all n, and compute

$$S_{n+1} = S_n + G_{n+1}$$

$$
\begin{aligned}
&= \frac{n(n+1)(2n+4)}{12} + \frac{(n+1)(n+2)}{2} \\
&= \frac{n(n+1)(2n+4) + 6(n+1)(n+2)}{12} \\
&= \frac{(n+1)[n(2n+4) + 6(n+2)]}{12} \\
&= \frac{(n+1)(2n^2 + 4n + 6n + 12}{12} \\
&= \frac{(n+1)(2n^2 + 10n + 12)}{12} \\
&= \frac{(n+1)(n+2)(n+3)}{6},
\end{aligned}
$$

which is exactly what the inductive hypothesis would predict for S_{n+1}.

Finally, then, we conclude that by the twelfth day of Christmas, the singer had received $S_{12} = (12 \times 13 \times 28)/12 = 13 \times 26 = 364$ gifts. If Christmas lasted all year, he or she would receive $S_{365} = 8,171,255$ gifts.